

Introduction to Hyperbolic Geometry

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Abstract__ Hyperbolic Geometry is a particular type of Non-Euclidean Geometry. We can see Hyperbolic Geometry and its application in our life . In this seminar we studied a brief history of Euclidean and Hyperbolic geometries, and discussed postulates of Euclidean Geometry . After that we introduced some definitions, axioms of Hyperbolic Geometry and main theorems in Neutral Geometry . Finally we talked about two models of Hyperbolic Geometry; Klein Disk Model and Poincaré Disk Model.

Keywords-component; Hyperbolic; Euclidean; Neutral Geometry.

I. INTRODUCTION

In general, when one refers to geometry, he or she is referring to Euclidean Geometry. Euclidean Geometry is the geometry with which most people are familiar. It is the geometry taught in elementary and secondary school. Euclidean geometry can be attributed to the Greek mathematician Euclid. His work entitled *The Elements* was the first to systematically discuss geometry. Since approximately 600B.C., mathematicians have used logical reasoning to deduce mathematical ideas, and Euclid was no exception. In his book, he started by assuming a small set of axioms and definitions, and was able to prove many other theorems. Although many of his results had been stated by earlier Greek mathematicians, Euclid was the first to show how everything fit together to form a deductive and logical system.

In mathematics, geometry is generally classified into two types, Euclidean and non-Euclidean one. The essential difference between Euclidean and Non-Euclidean geometry is the nature of parallel lines. Recall, that Euclid started with small postulates, five to be precise. The first four of these postulates are very clear and concise.

The Euclidean Postulates

1. Any two points can be joined by a straight line.
2. Any straight line segment can be extended indefinitely in a straight line.

3. Given any straight line segment, a circle can be drawn having the segment as a radius and one endpoint as center.

4. All right angles are congruent.

5. If a straight line falls on two straight lines in such a manner that the interior angles on the same side are together less than two right angles, the straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

Although fifth postulate has since been stated in simpler forms (Through a point not on a given straight line, one and only one line can be drawn that never meets the given line[Hilbert's parallel postulate]).

Mathematicians were still troubled by its complexity, and were convinced that it could be proven as a theorem using the other four postulates . But then many tried to find a proof by contradiction. For example, Girolamo Saccheri and Johann Lambert both tried to prove the fifth postulate by assuming it was false and looking for a contradiction. Lambert worked to further Saccheri's work by looking at a quadrilateral with three right angles.

Lambert looked at three possibilities for the fourth angle: right, obtuse or acute. He figured that if he could disprove that it is an obtuse or acute angle then he would prove the Hilbert's parallel axiom. However, Lambert was unable to disprove that the angle was acute. Carl Gauss, Nikolai Lobachevsky and Janos Bolyai, whose work focused on a negation of the Hilbert's parallel postulate, worked at approximately the same time. They assumed that there were at least two lines parallel or no lines parallel to the original line. These two tactics gave rise to two new geometries, Hyperbolic Geometry (at least two lines parallel) and Elliptic Geometry (no lines parallel). At the time of these discoveries, it was believed that these two geometries were false, and many people refused to believe that another geometry, other than Euclidean, existed.

In this seminar we will focus on one type of non Euclidean Geometry which is Hyperbolic Geometry, it was created in the first half of the nineteenth century in

the midst of attempts to understand Euclid's axiomatic basis for geometry.

" Hyperbolic Geometry is, defined by, the geometry you get by assuming all the axioms for Neutral Geometry¹ and replacing Hilbert's parallel postulate by one of its negation, which we shall call the 'Hyperbolic Parallel Axiom' " .

Returning to the definition of Hyperbolic Geometry, two parts were emphasized. First, the axioms of Neutral Geometry, which consist of the theorems that can be proven without the use of the fifth postulate, are included in Hyperbolic Geometry. Second, Hyperbolic Geometry includes a negation of the Hilbert's parallel postulate, the Hyperbolic Parallel Axiom which states that" in Hyperbolic Geometry there exist a line l and a point P not on l such that at least two distinct lines parallel to l pass through P " .

Hyperbolic Geometry appears clearly in cosmos. For example, the orbit of a planet is an ellipse with the sun as one of its focal points. It will remain an ellipse as long as it remains a closed curve, according to the laws of gravitation. If the planet were to speed up, the curve would open up into a parabola, and then a hyperbola. An object that passes through the solar system and does not orbit is on a hyperbolic path. The path of a comet as it passes the earth is a hyperbola. In addition ,this geometry has many application ,for example, in radar tracking stations: an object is located by sending out sound waves from two point sources: the concentric circles of these sound waves intersect in hyperbolas. Also in physics, for example, when you hold the temperature of a body of gas constant, then the volume and pressure are inversely related. If you plot those against each other, you find a hyperbola.

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II. HYPERBOLIC GEOMETRY

In this chapter, we will examine the axioms that form the basis for Hyperbolic Geometry. Next, we will examine the Neutral theorems .

Axioms of Hyperbolic Geometry:

According to Ramsey and Richter (1995) there are seven axioms in Hyperbolic Geometry:

Axiom 1: If A and B are distinct points then there is only one line that passes through both of them.

Axiom 2: Let M , a metric space, be a non-empty set. Then a distance metric d is defined on M for all pairs of points P and Q so that the following hold:

- $d(P, Q) \geq 0$ for all $P, Q \in M$.
- $d(P, Q) = 0$ if and only if $P = Q$.
- $d(P, Q) = d(Q, P)$ for all $P, Q \in M$.
- $d(P, R) \leq d(P, Q) + d(Q, R)$ for all $P, Q, R \in M$.

Axiom 3: For each line l , there is a 1-1 mapping, x , from the set l to set of real numbers, such that if A and B are any points on l , then

$$d(A, B) = |x(A) - x(B)|.$$

Axiom 4: If l is any line, there are three corresponding subsets HP_1 and HP_2 , called half-planes, and the line l such that the sets l , HP_1 and HP_2 are disjoint and their union is all of the plane P , such that:

- if P and Q are in the same half-plane then segment PQ contains no point of l .
- if $P \in HP_1$ and $Q \in HP_2$ then segment PQ contains a point of l .

Axiom 5: For each $\overrightarrow{h}, \overrightarrow{k}$ there is a number $\angle \overrightarrow{h}, \overrightarrow{k}^{\text{rad}}$, in the interval $[0, \pi]$ called the radian measure of the angle, such that:

- if $\overrightarrow{h}$ and $\overrightarrow{k}$ are the same ray then $\angle \overrightarrow{h}, \overrightarrow{k}^{\text{rad}} = 0$, if $\overrightarrow{h}$ and $\overrightarrow{k}$ are opposite rays then $\angle \overrightarrow{h}, \overrightarrow{k}^{\text{rad}} = \pi$.
- sum of the measure of the angle and its supplement equal π .
- If $\overrightarrow{j}$ is in the interior of an $\angle \overrightarrow{h}, \overrightarrow{k}$ then $\angle \overrightarrow{h}, \overrightarrow{j}^{\text{rad}} + \angle \overrightarrow{j}, \overrightarrow{k}^{\text{rad}} = \angle \overrightarrow{h}, \overrightarrow{k}^{\text{rad}}$.
- if a ray $\overrightarrow{k}$ from a point Z lies in a line l , then in each half-plane bounded by l , the set of rays $\overrightarrow{j}$ from Z is in a 1-1 correspondence with the set of real numbers α in $(0, \pi)$ in such a way that α is equal to the $\angle \overrightarrow{j}, \overrightarrow{k}^{\text{rad}}$ of $\angle \overrightarrow{j}, \overrightarrow{k}$.
- if the ray $\overrightarrow{j}$ is determined as $\overrightarrow{ZP}$, where P is a point on $\overrightarrow{j}$ then angle α depends continuously on P .

Axiom 6: If two sides and the included angle of a first triangle are congruent respectively to two sides and the included angle of a second triangle, then the triangles are congruent.

Axiom 7: There exists a line and a point not on the line such that there are at least two lines through that point that do not intersect the given line.

The first six axioms are the same in both Euclidean and Hyperbolic Geometry. The seventh one, more commonly known as the Hyperbolic Parallel Axiom, is the only axiom which differs from that of Euclidean Geometry.

The negation of the seventh axiom leads to various consequences and thus differences between Euclidean and Hyperbolic geometries. In Euclidean Geometry, there is only one line through a point not on a line that is parallel to the given line. Hyperbolic Geometry is based on one of the negation of that postulate. In Hyperbolic Geometry there are at least two lines that are parallel to the given

¹Neutral Geometry is the geometry derived from the first four postulates of Euclid and it's theorems are true in both Euclidean and Hyperbolic geometries.

line. Another subtle difference is in the sum of the angles in a triangle. According to Ramsey and Richter, “the sum of the three interior angles of a triangle is less than or equal to 180° ”. They later go on to explain that in Euclidean Geometry the sum of the measures of the angles of a triangle equals 180 degrees while in Hyperbolic Geometry the sum of the measures of the angles of a triangle is less than 180 degrees .

The axioms that form the basis of Hyperbolic Geometry have been given. The Neutral theorems will now be examined. The theorems are called Neutral because the theorems are the same in both Euclidean and Hyperbolic geometries. These theorems can be proved without the Hilbert’s parallel postulate .

Neutral Geometry

Main theorems in Neutral Geometry:

Theorem 1: All vertical angles are congruent.

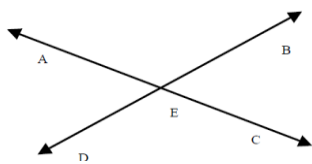


Figure 1 vertical lines

Proof:

Let line AC and line BD intersect at point E. Therefore, $\angle AEB$ is supplementary to $\angle AED$ and $\angle BEC$. Supplement angles add to 180° by Axiom 5. Since $\angle AED$ and $\angle BEC$ have the same supplement then they are congruent. Therefore vertical angles are congruent.

Theorem 2: An exterior angle of a triangle is greater than each of its nonadjacent interior angles.

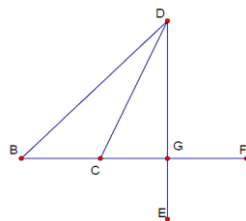


Figure 2

(The proof is omitted).

Theorem 3: If two angles and the included side of a triangle are congruent to two angles and the included

side of a second triangle then the triangles are congruent.

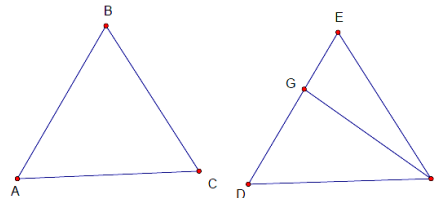


Figure 3

Proof:

Referring to Figure 3, $\triangle ABC$ and $\triangle DEF$ are triangles with $AC \cong DF$, $\angle BCA \cong \angle EFD$ and $\angle BAC \cong \angle EDF$. Plot G on ED so that $AB \cong GD$, then $\triangle ABC \cong \triangle DGF$ by axiom 6, Side-Angle-Side.

Therefore, $\angle BCA \cong \angle GFD$ because corresponding parts of congruent triangles are congruent. So $\angle GFD \cong \angle DFE$ by the transitive property of equality. Since those angles are congruent then GF coincides with EF by axiom 5. So, $\triangle ABC \cong \triangle DEF$.

Therefore it can be proven that triangles are congruent by Angle-Side-Angle .

Theorem 4: If two angles and a non-included side of one triangle are congruent to two angles and a non-included side of another triangle ,then the two triangles are congruent.

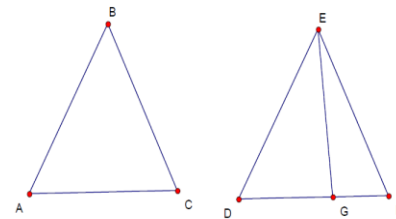


Figure 4

Proof:

Referring to Figure 4, $\triangle ABC$ and $\triangle DEF$ are triangles with $AB \cong DE$, $\angle BAC \cong \angle EDF$ and $\angle ACB \cong \angle DFE$. Plot G on DF so that $AC \cong DG$. Then $\triangle ABC \cong \triangle DEG$ by Side-Angle-Side. Therefore, $\angle BCA \cong \angle EGD$ because corresponding parts of congruent triangles are congruent. So, $\angle EGD \cong \angle EFG$ by the transitive property of equality. This means that GE and EF would not meet because if the corresponding angles of two lines are congruent then the lines are parallel lines, which are lines that do not intersect. This is a contradiction to the assumption that they do meet. So, G and F must coincide. Therefore, triangles can be proven congruent by Angle-Angle-Side.

Theorem 5: If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse leg of another right triangle then the triangles are congruent.

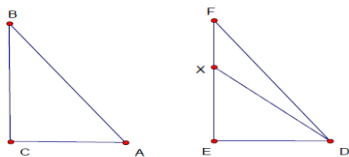


Figure 5

(The proof is omitted).

Theorem 6: If a triangle is equilateral then it is equiangular.

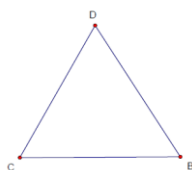


Figure 6

Proof:

Let $\triangle CDB$ be an equilateral triangle. Then $\triangle CDB$ is an isosceles triangle with $DC \cong DB$. Therefore the base angles are congruent. So $\angle DCB \cong \angle DBC$. Also, $\triangle BDC$ is also an isosceles triangle with $BD \cong BC$. Since the base angles are congruent, then $\angle BDC \cong \angle BCD$. Therefore $\angle DBC \cong \angle DCB \cong \angle BDC$. This proves that if you have an equilateral triangle then you have an equiangular triangle.

Theorem 7: The longest side of a triangle is across from the greatest angle.

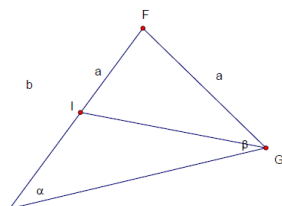


Figure 7

Proof:

Referring to Figure 7, $\triangle FGH$ is a triangle with $b = FH$ and $a = FG$. Let $\alpha = \angle FHG$ and $\beta = \angle FGH$ and assume $a < b$. Find I on FH so that $IF = FG$. So, $\triangle FIG$ is an isosceles triangle, which means that $\angle FIG$ and $\angle FGI$ are congruent. Therefore $\angle FGH$ is greater than $\angle FGI$ by axiom 5. Then $\angle FIG$ is greater than $\angle FHG$ by Theorem 2. Therefore, $\angle FGH$ is greater than $\angle FHG$ that is $\beta > \alpha$. Based on the assumption that $a < b$, the proven result that $\beta > \alpha$ and that β is opposite b and α is opposite a , the longest side of a triangle is opposite the greatest angle.

Theorem 8: If three sides of one triangle are congruent to the corresponding three sides of another triangle, then the triangles are congruent.

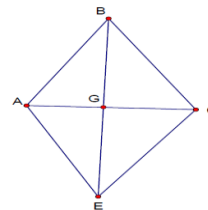


Figure 8

Proof:

Referring to Figure 8, $\triangle ABC$ and $\triangle AEC$ are triangles with $AB \cong AE$, $BC \cong EC$ and $AC \cong AC$. Then $\triangle BCE$ and $\triangle BAE$ are isosceles triangles.

Therefore, $\angle CBG \cong \angle CEG$ and $\angle ABG \cong \angle AEG$ because of Theorem 6. So, $\angle AEC \cong \angle ABC$ by axiom 5.

Therefore, $\triangle ABC \cong \triangle AEC$ by Side- Angle- Side.

The proofs of the Neutral theorems are identical to proofs of the analogous results in Euclidean Geometry. The proofs are identical because the proofs do not use the Hilbert's parallel postulate. The next theorems and their proofs may or may not be affected by the Hyperbolic Parallel Axiom. Therefore, the proofs may differ from those in Euclidean Geometry.

Theorems That Might Be Affected By the Hyperbolic Parallel Axiom

The study of quadrilaterals in Hyperbolic Geometry is one instance in which Hyperbolic Geometry differs from Euclidean Geometry. Rectangles, quadrilaterals having four right angles, do not exist in Hyperbolic Space. Quadrilaterals can be divided into two triangles, $\triangle ABD$ and $\triangle BCD$ by drawing one diagonal.

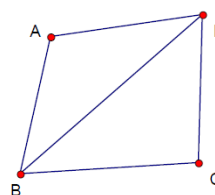


Figure 9

$$\angle ABD + \angle BDA + \angle DAB < 180^\circ$$

$$\angle DBC + \angle BCD + \angle CDB < 180^\circ$$

By angle addition:

$$\angle ABD + \angle DBC = \angle ABC$$

$$\angle BDA + \angle CDB = \angle CDA$$

By adding the two inequalities:

$$\angle ABC + \angle BCD + \angle CDA + \angle DAB < 360^\circ$$

If three of the angles were 90 degrees then by the argument above, the last angle would be less than 90

degrees. Euclidean rectangles are quadrilaterals with four right angles. The sum of the angles of a rectangle equals 360 degrees. Therefore, rectangles do not exist in Hyperbolic Geometry. Therefore, squares, quadrilaterals having four right angles and four congruent sides, do not exist in Hyperbolic Geometry. This, however, does not mean that there are no regular quadrilaterals in Hyperbolic Geometry. A regular quadrilateral is one for which all sides are congruent and that all angles are congruent. Regular quadrilaterals and rhombus exist in Hyperbolic Geometry because there is a model in which such figures exist. "The main property of any model of an axiom system is that all theorems of the system are correct in the model. This is because logical consequences of correct statements are themselves correct". These next proofs focus on the characteristics of rhombus and regular quadrilaterals in Hyperbolic Geometry.

Theorem 9: The diagonals of a rhombus bisect each other, are perpendicular, and bisect the angles of the rhombus.

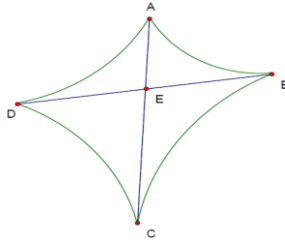


Figure 10

Proof:

Quadrilateral ABCD is a rhombus with all sides congruent and diagonals AC and DB. So, $\triangle ABD \cong \triangle CBD$ and $\triangle ADC \cong \triangle ABC$ by Side-Side-Side. Therefore, $\angle DAB \cong \angle DCB$ and $\angle ADC \cong \angle ABC$. That is, the opposite angles of a rhombus are congruent. Continuing with the proof $\angle DAE \cong \angle BAE$ and $\angle ADE \cong \angle CDE$ since corresponding parts of congruent triangles are congruent. So, $\triangle DAE \cong \triangle BAE$ and $\triangle ADE \cong \triangle CDE$ by Side-Angle-Side. So, $AE \cong EC$ and $DE \cong EB$. Therefore, the diagonals of a rhombus bisect each other. Continuing further, $\angle AEB \cong \angle AED$ since all corresponding parts of congruent triangles are congruent. These angles are also supplements since the angles share a side and the other two sides are opposite rays and therefore the measures sum to π .

Therefore, the diagonals of a rhombus are perpendicular. Continuing, $\triangle AEB \cong \triangle CEB$ and $\triangle DEC \cong \triangle BEC$ by Side-Side-Side. So $\angle ABE \cong \angle CBE$ and $\angle DCE \cong \angle BCE$. Therefore, the diagonals of a rhombus bisect the angles of the rhombus.

Theorem 10: The diagonals of a regular quadrilateral bisect each other and they are perpendicular to each other.

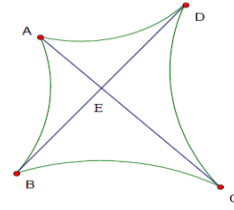


Figure 11

Regular quadrilaterals are a subset of rhombus. Therefore Theorem 10 has already been proven through Theorem 9.

III. HYPERBOLIC GEOMETRY MODELS

So far, we have developed Hyperbolic Geometry axiomatically, that is independent of the interpretation of the words 'point' and 'line'. To help visualize objects within the geometry, and to make calculations more convenient we use a model. We define points and lines as certain 'idealized' physical objects that are consistent with the axioms. This system of lines and points is the model of the geometry. Though the pictures drawn in the model are consistent with the axiomatic development of the geometry it represents, they are not the geometry, merely a way of picturing objects and operations within the geometry. Probably the best known model of a geometry is:

The Euclidean Model

This model is derived by defining a point to be an ordered pair of real numbers (x, y) ,

a line to be the sets of ordered pairs (points) that solve an equation having the form

$ax + by = c$ where a, b and c are given real numbers, and the plane to be the collection of all points. Two lines $ax + by = c$ and $dx + ey = f$ are said to intersect if there exists a point (x, y) that satisfies both equations.

The distance between two points $A(x, y)$ and $B(z, w)$ in the plane is given by:

$$d(A, B) = \sqrt{(z - x)^2 + (w - y)^2}$$

And the angle between two lines $ax + by = c$ and $dx + ey = f$ by:

$$\text{angle} = \left| \tan^{-1} \left(\frac{-d}{e} \right) - \tan^{-1} \left(\frac{-a}{b} \right) \right|$$

(or by π minus this value.)

This model is consistent with the five postulates of Euclidean Geometry, and is usually referred to as the Euclidean Plane, the Real Plane, or R^2 . It is assumed that the reader is familiar with the Euclidean Plane Model, and we will move on to the hyperbolic models. There are three commonly used models for Hyperbolic Geometry that define a real Hyperbolic Space which satisfies the axioms of Hyperbolic Geometry. These models are the

Klein Disk, the Poincaré Disk and The Poincaré Upper Half-Plane models. All three are realized with the Euclidean Plane, but all three have entirely different flavors, especially when constructing objects within them. All three also have their advantages and disadvantages. We will discuss Klein Disk Model and Poincaré Disk Model.

The Klein Disk Model (KDM)

For the actual definition and construction of the most basic objects such as points and lines, the Klein Disk Model is the easiest of the three. For this reason we introduce it first. For anything more complicated, such as calculating angle measures, it is considerably less convenient.

When introducing parallel lines to middle school or high school students, teachers often say something along the lines of, “Parallel lines never meet no matter how far you extend them. Lines that are not parallel will eventually meet if you extend them far enough.” The Klein Disk Model, (or KDM) removes this distinction by eliminating the infinitude (in the Euclidean sense) of the line.

The model consists of the interior of the unit circle. The points are Euclidean points within the unit circle $\{(x,y) : x^2 + y^2 < 1\}$, ideal points lie on the unit circle $\{(x,y) : x^2 + y^2 = 1\}$, and ultra-ideal points lie out the unit circle $\{(x,y) : x^2 + y^2 > 1\}$. The lines are the portions of Euclidean lines lying within the unit circle, or the chords of the circle, and two lines intersect if they intersect in the Euclidean sense and the point of intersection lies inside the unit circle.

(Figure 12) illustrates the model. A, B, C, D and O are points; P, Q, R and S are ideal points; and AB, CD, OC, and CP are lines. Notice that line AB may also be referred to as AP, BQ, PQ or any combination of two distinct points or ideal points lying on it.

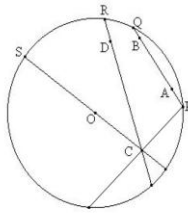


Figure 12

Note that line AB is limiting parallel to line CP, and divergently parallel to CD and CO where:

- limiting parallel:
Two lines are said to be limiting parallel if have no common points within the model whenever they intersect on the boundary.
- divergently parallel:

Two lines are said to be divergently parallel if they share no common points within the model or on the model's boundary.

A tool that we will be using in the discussions of metric in all three models is the cross ratio. For that reason we will introduce it here. Given four points in the plane, A, B, P and Q, we define the cross ratio (AB,PQ) by:

$$(AB, PQ) = \frac{(AP)(BQ)}{(AQ)(BP)}$$

where, e.g., AP is the length of the Euclidean segment AP.

The metric of KDM

The distance between any two points A to B in KDM is defined as follows:

$$h(A, B) = \frac{1}{2} \left| \ln \left(\frac{(AP)(BQ)}{(AQ)(BP)} \right) \right| = \frac{1}{2} |\ln (AB, PQ)|$$

where P and Q are the ideal points associated with line AB.

If A and B coincide, then $(AB, PQ) = 1$, and $h(A, B) = 0$, so $h(A, A) = 0$.

The cross ratios (AB, PQ) and (BA, PQ) are merely reciprocals of each other, so the absolute values of the logs of these expressions will be equal, and $h(A, B) = h(B, A)$.

We will not show the triangle inequality for the metric, but we can confirm easily

that $h(A, B) + h(B, C) = h(A, C)$ if A, B and C are collinear:

$$\begin{aligned} h(A, B) + h(B, C) &= \frac{1}{2} |\ln (AB, PQ)| + \frac{1}{2} |\ln (BC, PQ)| \\ &= \frac{1}{2} \left| \ln \left(\frac{(AP)(BQ)}{(AQ)(BP)} \right) \right| + \frac{1}{2} \left| \ln \left(\frac{(BP)(CQ)}{(BQ)(CP)} \right) \right| \\ &= \frac{1}{2} \left| \ln \left(\frac{AP \cdot BQ}{AQ \cdot BP} \cdot \frac{BP \cdot CQ}{BQ \cdot CP} \right) \right| \\ &= \frac{1}{2} \left| \ln \left(\frac{AP \cdot CQ}{AQ \cdot CP} \right) \right| = h(A, C) \end{aligned}$$

Notice that as A and B become very close to each other (AB, PQ) approaches 1 and the metric approaches zero. Notice also that as A (or B) approaches P (or Q) the cross ratio (AB, PQ) approaches either zero or infinity, and $h(A, B)$ approaches infinity. So, with this metric, our lines are indeed infinite.

Angle measure in KDM

A disadvantage of KDM is that it does not represent angles ‘accurately’, in fact the definition of angle measure is rather inconvenient. For lines l and m intersecting in point A, we define the measure of the angle formed by l and m at A as the angle formed by l' and m' at A' where l' and m' are the arcs of circles orthogonal to the unit circle at the endpoints of l and m, and A' is the intersection of l' and m'. This is illustrated in (Figure 13).

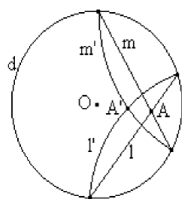


Figure 13

This angle measure gives us a curious definition for perpendicularity in KDM. In KDM, each line has associated with it an ultra-ideal point exterior to d (the unit circle) called the polar point of the line. It is defined for line l in KDM as the intersection L of the lines tangent to d at the endpoints of l. (Figure 14) A line through O will have no polar point. (We can think of it as having its polar point at infinity).

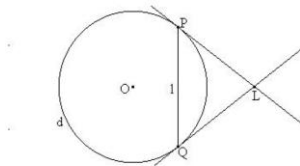


Figure 14

We define a line m as perpendicular to line l if the extension of line m contains the polar point L of line l, (l will contain M) (Figure 15).

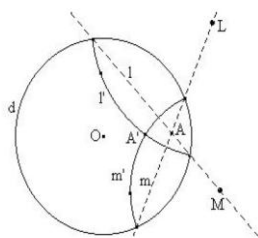


Figure 15

One nice thing about KDM is that it has rotational symmetry, so regular polygons and tessellations have a pleasing and complete appearance that reminds one of, and may well have inspired, some of the works of M.C. Escher. (Figure 16) depicts a partial tessellations of KDM by equilateral triangles.

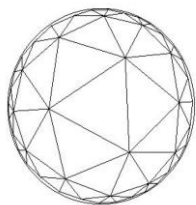


Figure 16

The Poincaré Disk Model (PDM)

The second model we will consider is the Poincaré Disk Model, or PDM. It is somewhat similar to KDM in appearance. The slightly more complicated definition of lines in PDM gives it an important advantage over KDM. It is conformal.

PDM also resides in the interior of the unit circle d in the Euclidean Plane. As in KDM, the points of PDM are the points lying interior to d, ideal points lie on d, and ultra ideal points lie exterior to d. The lines of PDM are general Euclidean circles (Euclidean lines and circles) orthogonal to d. These will either be arcs of Euclidean circles orthogonal to d (line AB in Figure 17), or diameters of d (line OC in Figure 17). Note that (Figure 17) shows the same situation for PDM as was shown for KDM in (Figure 12).

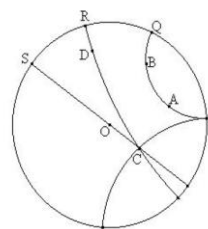


Figure 17

The Metric of PDM

The metric in PDM is the same as in KDM:

$$h(A, B) = \frac{1}{2} \left| \ln \left(\frac{(AP)(BQ)}{(AQ)(BP)} \right) \right| = \frac{1}{2} |\ln (AB, PQ)|$$

where P and Q are the ideal points at the 'ends' of the line AB. All the same properties of the metric hold.

Angle Measure in PDM

The measure of the angle formed at point A by lines l and m is defined as the measure of the angle formed by lines l' and m' at A where l' and m' are the Euclidean lines tangent to l and m, respectively, at A.

The polar points of our lines in PDM (defined the same way as in KDM) make calculating angle measure simple. The angle formed by lines l and m at point A is equal to the measure of angle LAM, or its complement, where L and M are the polar points of l and m respectively. (Figure 18) It is evident that rotation through a right angle about A sends the tangents to the Euclidean circles l and m at A to the lines LA and MA, which are the radii of the Euclidean circles l and m.

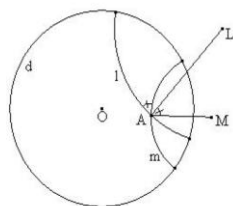


Figure 18

While constructions in PDM tend to be more complicated than in KDM, the fact that PDM is conformal makes the pictures of objects look more like they 'should'. For example, (Figure 19) shows a right triangle in both KDM and PDM. The right angle at C looks right in PDM, but not in KDM.

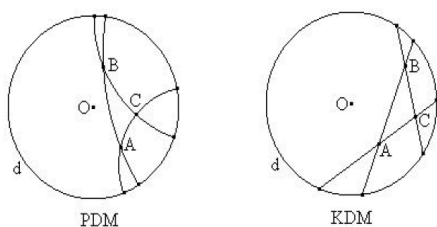


Figure 19

Tessellations are also symmetric and nice in PDM. (Figure 20) shows a partial tiling of the plane by equilateral triangles.

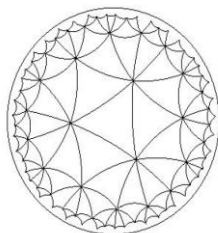


Figure 20

While both KDM and PDM allow for easy visualization, a major disadvantage of both is that any calculations are tedious and messy.

REFERENCES

- [1] C. Donald. "Hyperbolic Geometry in the High School Geometry Classroom," LOWA STATE UNIVERSITY . www.math.iastate.edu .
- [2] H. Wolfe. "NON-EUCLIDEAN GEOMETRY," The Dryden Press, 1945.
- [3] J. Cannon, W. Floyd, R. Kenyon, and W. Parry. "Hyperbolic Geometry," MSRI Publications, 1997.
- [4] L. Christina. "HYPERBOLIC GEOMETRY," Sheets. UNIVERSITY OF NEBRASKA- LINCOLN. www.unl.edu. July 2007.
- [5] S. Ross. "NON-EUCLIDEAN GEOMETRY," Master thesis, B.S. University of Maine, 1990.