# Super linear algebra

## Block matrices

Loai M.Shaalan , Nureddin Rabie
Department of mathematics
Palestine polytechnic university
Hebron ,Palestine .
Loay_101@yahoo.com

***Abstract* - This seminar has three parts . Part one introduce the notion of super matrices . Part two discuss the notions of super linear algebra , the characteristic super polynomial of a linear transformation , the minimal polynomial , and Cayley–Hamilton theorem . In part three we have an application of using super matrices for solving BVP .**

***Keywords :Super matrices ;Super linear algebra ; Block matrices .***

PART ONE

INTRODUCTION

In this part we shall explain the notion of super matrices , super vector space and give some examples on it .

* Super matrices

<u>Definition(1.1)</u>:A super matrix is a matrix whose entries are matrices .

<u>Example(1.1)</u>: Let
$$A=\left[\begin{array}{cc|ccc} 0 & 1 & 7 & 2 & 5 \\ 3 & 5 & 16 & 2 & 4 \\ \hline 6 & 8 & 0 & 5 & 3 \\ 9 & 9 & 7 & 2 & 3 \\ 7 & 0 & 5 & 2 & 4 \end{array}\right]$$

Then A is a super matrix with two rows and two columns .

Now let

$$A_{11}=\begin{bmatrix} 0 & 1 \\ 3 & 5 \end{bmatrix}, A_{12}=\begin{bmatrix} 7 & 2 & 5 \\ 16 & 2 & 4 \end{bmatrix}, A_{21}=\begin{bmatrix} 6 & 8 \\ 9 & 9 \\ 7 & 0 \end{bmatrix}, A_{22}=\begin{bmatrix} 0 & 5 & 3 \\ 7 & 2 & 3 \\ 5 & 2 & 4 \end{bmatrix}$$

Then we can write

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

<u>Note</u> :Sometimes a super matrix is called a block matrix .And we call the entries sub matrices or blocks .

<u>Definition(1.2)</u>:A super matrix of the form

$$D=\begin{bmatrix} D_{11} & 0 & & 0 \\ 0 & D_{22} & & 0 \\ & & & \\ & & & \\ 0 & 0 & & D_{nn} \end{bmatrix}$$ is called a super diagonal matrix .

<u>Example(1.2)</u>: The super matrix

$$A=\left[\begin{array}{cc|ccc|ccc} 5 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 7 & 6 & 5 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 2 & 5 \end{array}\right]$$ is a super diagonal matrix .

<u>Definition(1.3)</u>:A super matrix D where $D_{ii}$ are square matrices is called a symmetrically partitioned super matrix .

Example(1.3): The super matrix

$$A=\left[\begin{array}{cc|ccc|c} 5 & 4 & 1 & 3 & 4 & 0 \\ 3 & 2 & 6 & 8 & 8 & 8 \\ \hline 5 & 6 & 6 & 8 & 0 & 6 \\ 3 & 8 & 5 & 8 & 1 & 5 \\ 4 & 4 & 4 & 8 & 0 & 4 \\ \hline 5 & 0 & 1 & 7 & 9 & 8 \end{array}\right]$$ is a

symmetrically partitioned super matrix .

## PART TWO

## SUPER LINEAR ALGEBRA

In this part we shall define the notion of super linear algebra and give some examples on it; also we shall discuss the characteristic super polynomial of a super matrix and Cayley-Hamilton theorem .

Definition(2.1) : Let V be a super vector space over the field F .We say V is a super linear algebra over F if and only if for every pair of super vectors α , β the product of α and β denoted by αβ is defined in V in such a way that :

- $\alpha(\beta\gamma)=(\alpha\beta)\gamma \qquad \forall \alpha,\beta,\gamma\in V$ .
- $(\alpha+\beta)\gamma=\alpha\gamma+\beta\gamma$ and $\gamma(\alpha+\beta)=\gamma\alpha+\gamma\beta \quad \forall \alpha,\beta,\gamma\in V$ .
- $c(\alpha\beta)=(c\alpha)\beta=\alpha(c\beta)$ $\forall \alpha,\beta\in V, c\in F$ .

Definition (2.2) : If a super linear algebra $V$ has an element $I$ such that $Ia=aI, \forall a\in V$ , then we call $V$ a super linear algebra with identity .

Definition (2.3) : A super linear algebra $V$ is called commutative if $ab=ba \quad \forall a,b\in V$ .

Example (2.1) : Let

$$V=\{[x_1 \quad x_2 \mid x_3 \quad x_4 \mid x_5], x_i\in Q, 1\le i\le 5\}$$

be a super vector space over $R$ .Define for $\alpha,\beta\in V$ ;

$$\alpha=[x_1 \quad x_2 \mid x_3 \quad x_4 \mid x_5]$$

and $\beta=[y_1 \quad y_2 \mid y_3 \quad y_4 \mid y_5]$

$$\alpha\beta=[x_1y_1 \quad x_2y_2 \mid x_3y_3 \quad x_4y_4 \mid x_5y_5]$$

Where $x_i, y_i\in R; 1\le i\le 5$.

Then $V$ is a super linear algebra over $R$ , and it's a commutative super linear algebra with unity $[1 \quad 1 \mid 1 \quad 1 \mid 1]$.

Example (2.2) : Let

$$V=\left\{\left[\begin{array}{cc|c} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ \hline x_7 & x_8 & x_9 \end{array}\right] x_i\in Q, 1\le i\le 9\right\}$$

then $V$ is a super linear algebra over $Q$ with the usual multiplication of matrices , and it's clear that $V$ is a non-commutative super linear algebra with unity $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$ .

Example (2.3) : Let $V$ be a super vector space over $F$ then $SL(V,V)$ is a super linear algebra over $F$ , under the operation of functions composition .

Definition (2.3) : Let

$$V=\left\{\begin{array}{l} [A_1 \; A_2 \cdots A_k]: A_i \text{ are row vectors} \\ \text{with entries from a field } F \end{array}\right\}$$

and let $T\in SL(V,V)$ , $T=[T_1\, T_2 \cdots T_k]$.

The k-tuple $c=(c_1,c_2,\dots,c_k)$ is called a characteristic super value of $T$ if there is a non-zero super vector $a \in V$ such that :

$$T(a) = ca$$
$$= (c_1, c_2, \ldots, c_k)[a_1\ a_2\ \ldots\ a_k]$$
$$= [c_1 a_1\ c_2 a_2\ \ldots\ c_k a_k]$$

And we call $a$ a characteristic super vector of $T$ associated with $c = (c_1, c_2, \ldots, c_k)$ .

Definition (2.4) : Let A be a square super diagonal matrix whose diagonal matrices are also squares , then the super determinant of A is defined as :

$$|A| = \begin{bmatrix} |A_1| & \cdots & \cdots & \\ \vdots & |A_2| & \cdots & \\ \vdots & \vdots & \ddots & \\ & & & |A_n| \end{bmatrix} = [|A_1| \quad |A_2| \quad \cdots \quad |A_n|]$$

where $|A_i|$ is the determinant of $A_i$ .

$i = 1, 2, \ldots\ldots\ldots, n$ .

Note : if A is a square super diagonal matrix whose diagonal matrices are also squares , then the super determinant of A is a super row vector .

Example (2.4) : Find the determinant of A where :

$$A = \left[\begin{array}{cc|cc|cc|c} 7 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 5 & 8 & 0 & 0 & 0 \\ 0 & 0 & 4 & 6 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{array}\right]$$

Solution :

$$|A| = \left[\begin{vmatrix} 7 & 3 \\ 1 & 1 \end{vmatrix} \quad \begin{vmatrix} 5 & 8 \\ 4 & 6 \end{vmatrix} \quad \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} \quad |7| \right]$$
$$= [4 \mid -2 \mid 0 \mid 7]$$

Definition (2.5) : A super polynomial is a polynomial of the form :

$$P(X) = [P_1(X) \quad P_2(X) \quad \ldots \quad P_n(X)];$$

$p_i(x)$ is a polynomial $1 \le i \le n$.

Example (2.5) : Let

$V = \{[R^2(X) \quad R^7(X) \quad R^1(X)]\}$ $R^i(X)$ is a polynomial of degree less than or equal i with coefficients from the real numbers ,then V is a super vector space over R with the operation of addition and scalar multiplication , ( the dimension of V is 3+8+2=13 ) .It's clear that V is a super linear algebra over R if we define for $P_1(X), P_2(X) \in P(X)$ and

$$P_1(X) = [P_1^2(X) \quad P_1^7(X) \quad P_1^1(X)]$$
$$P_2(X) = [P_2^2(X) \quad P_2^7(X) \quad P_2^1(X)]$$

$$P_1(X)P_2(X) = [P_1^2(X)P_2^2(X) \quad P_1^7(X)P_2^7(X) \quad P_1^1(X)P_2^1(X)]$$

Definition (2.6) : Let

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$$

be a square super diagonal square matrix with entries from a field F where $A_i$ is a square matrix $i = 1, 2, \ldots\ldots n$ . Then the super polynomial :

$$P(X) = \det(xI - A)$$
$$= [\det(xI - A_1) \quad \det(xI - A_2) \quad \cdots \quad \det(xI - A_n)]$$
$$= [f_1 \quad f_2 \quad \cdots \quad f_n]$$

is called the characteristic super polynomial (Super characteristic polynomial ) of A .

Note : The order of the super characteristic polynomial P(X) in the previous definition is $(k_1, k_2, \ldots, k_n)$ where $k_i$ is the order of the matrix $A_i$ , $1 \le i \le n$ .

Example (2.6) : Let

$$A = \left[\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ \hline 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right]$$ , then the characteristic super polynomial of A is

$$|xI - A| = \begin{bmatrix} \begin{vmatrix} x-2 & 1 \\ 3 & x-1 \end{vmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{vmatrix} x-4 & 2 \\ -2 & x-4 \end{vmatrix} \end{bmatrix}$$

$$= \left[ (x-2)(x-1) - 3 \;\middle|\; (x-4)^2 - 4 \right]$$

$$= \left[ x^2 - 3x - 1 \;\middle|\; x^2 - 8x - 12 \right]$$

To find the characteristic super value we must find the zeros of the characteristic super polynomial :

$$P(x) = [P_1(x) \quad P_2(x)]$$

$$P_1(x) = x^2 - 3x - 1$$

$$P_2(x) = x^2 - 8x - 12$$

If $P_1(x) = x^2 - 3x - 1 = 0$ then

$$x = \frac{3 \pm \sqrt{9+4}}{2} = \frac{3 \pm \sqrt{13}}{2}$$

And if $P_2(x) = x^2 - 8x - 12 = 0$ then

$$x = \frac{8 \pm \sqrt{64+48}}{2} = \frac{8 \pm \sqrt{112}}{2} = 4 \pm 2\sqrt{7}$$

So the characteristic super values are :

$$C_1 = \left[ \frac{3+\sqrt{13}}{2} \;\middle|\; 4 + 2\sqrt{7} \right]$$

$$C_2 = \left[ \frac{3+\sqrt{13}}{2} \;\middle|\; 4 - 2\sqrt{7} \right]$$

$$C_3 = \left[ \frac{3-\sqrt{13}}{2} \;\middle|\; 4 + 2\sqrt{7} \right]$$

$$C_4 = \left[ \frac{3-\sqrt{13}}{2} \;\middle|\; 4 - 2\sqrt{7} \right]$$

Example(2.7): Let $A = \left[\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$

Then the super characteristic polynomial of $A$ is :

$$P(\lambda) = \left[ \lambda^2 - 4\lambda + 3 \;\middle|\; \lambda^2 - 2\lambda + 2 \right].$$

And if we solve $P(\lambda) = 0$ then the super eigenvalues are :

$$\lambda_1 = [1 \mid 1+i]$$
$$\lambda_2 = [3 \mid 1+i]$$

$$\lambda_3 = [1 \mid 1-i]$$
$$\lambda_4 = [3 \mid 1-i]$$

With the associated super eigenvectors :

$$V_1 = \left[\begin{array}{c} -1 \\ 1 \\ \hline i \\ 1 \end{array}\right] \qquad V_2 = \left[\begin{array}{c} 1 \\ 1 \\ \hline i \\ 1 \end{array}\right] \quad V_3 = \left[\begin{array}{c} -1 \\ 1 \\ \hline -i \\ 1 \end{array}\right] \; V_4 = \left[\begin{array}{c} 1 \\ 1 \\ \hline -i \\ 1 \end{array}\right]$$

Related to the super eigenvectors respectively .

Definition (2.7) : Let $T = [T_1 \quad T_2 \quad \cdots \quad T_n]$ be a linear operator on an infinite dimensional super vector space $V=[V_1 \quad V_2 \quad \cdots \quad V_n]$ of dimension $(n_1\ ,n_2\ ,\cdots,n_n\ )$ over the field $F$ , the minimal super polynomial for $T$ is the unique monic super generator of the super ideal of super polynomials over $F$ which super annihilate $T$ .

This definition means that the minimal super polynomial $P=[P_1(x) \quad P_2(x) \quad \cdots \quad P_n(x)]$ for the linear operator $T$ is uniquely determined by the following properties :

1) $P=[P_1 \quad P_2 \quad \cdots \quad P_n]$ is a monic polynomial over the field $F$ .
2) $\mathrm{P(T)}=[\mathrm{P_1(T_1)} \quad \mathrm{P_2(T_2)} \quad \cdots \quad \mathrm{P_n(T_n)}]$ $=[0 \quad 0 \quad \cdots \quad 0]$.
3) No super polynomial over $F$ which annihilate $T$ has smaller degree than $P$ .

In the case of square super diagonal square matrices $A$ , we define the minimal super polynomial for $A$ as the unique monic super generator of super ideal of all super polynomials over $F$ which super annihilate $A$ .

If the operator $T$ represented in some ordered super basis by the super square diagonal square matrix $A$ , then $T$ and $A$ have the same minimal super polynomial , that is because

$$f(T)=[f_1(T_1) \quad f_2(T_2) \quad \cdots \quad f_n(T_n)]$$

is represented by the matrix

$$f(A)=[f_1(A_1) \quad f_2(A_2) \quad \cdots \quad f_n(A_n)]$$

so that $f(T)=[0 \quad 0 \quad \cdots \quad 0]$ if and only if $f(A)=[0 \quad 0 \quad \cdots \quad 0]$

(i.e : $\mathrm{f_i(T_i)}=0$ if and only if $\mathrm{f_i(A_i)}=0$) .

<u>Theorem (3.1)</u> : Let $T$ be a linear operator on an $(n\ ,n\ ,\cdots,n_n)$ dimensional vector space $V=[V_1 \quad V_2 \quad \cdots \quad V_n]$ or ( let $A$ be an $(n_1\times n_1\ ,n_2\times n_2\ ,\cdots,n_n\times n_n)$ square super diagonal square matrix ). The characteristic super polynomial and minimal super polynomials for $T$ (for $A$ ) have the same super roots .

<u>Proof</u> : Let $P=[P_1 \quad P_2 \quad \cdots \quad P_n]$ be the minimal super polynomial for

$$\mathrm{T}=[\mathrm{T_1} \quad \mathrm{T_2} \quad \cdots \quad \mathrm{T_n}]$$

( $i,e:P_i$ is the minimal polynomial of $T_i\ ,\ i=1\ ,2\ ,\cdots,n$ ) .

Let $c=(c_1 \quad c_2 \quad \cdots \quad c_k)$ be a scalar . we want to prove

$$\mathrm{P(c)}=[\mathrm{P(c_1)} \quad \mathrm{P(c_2)} \quad \cdots \quad \mathrm{P(c_n)}]=[0 \quad 0 \quad \cdots \quad 0]$$

if and only if $\mathrm{c}$ is the characteristic super value for $T$ .

First we suppose

$$p(c)=[P_1(c_1) \quad P_2(c_2) \quad \cdots \quad P_n(c_n)] =[0 \quad 0 \quad \cdots \quad 0]$$

Then

$$P=[P_1 \quad P_2 \quad \cdots \quad P_n]=(x-c)q= [(x-c_1)q_1 \quad (x-c_2)q_2 \quad \cdots \quad (x-c_n)q_n]$$

where $q=[q_1 \quad q_2 \quad \cdots \quad q_n]$ is a super polynomial , since super degree of $q$ is less than the super degree of $P$ .

(i,e $\deg(\mathrm{q_i})<\deg(\mathrm{P_i})$ $\mathrm{i}=1\ ,2\ ,\cdots,\mathrm{n}$ ) .

the definition of the minimal super polynomial $P$ tells us that

$$q(T)=[q_1(T_1) \quad q_2(T_2) \quad \cdots \quad q_n(T_n)]\neq [0 \quad 0 \quad \cdots \quad 0]$$

Now choose a super vector $\beta=[\beta_1 \quad \beta_2 \quad \cdots \quad \beta_n]$ such that

$$q(T)\beta = [q_1(T_1)\beta_1 \quad q_2(T_2)\beta_2 \quad \cdots \quad q_n(T_n)\beta_n] =$$
$$[0 \quad 0 \quad \cdots \quad 0]$$

Now let

$$\alpha = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n] = q(T)\beta =$$
$$[q_1(T_1)\beta_1 \quad q_2(T_2)\beta_2 \quad \cdots \quad q_n(T_n)\beta_n]$$

Then

$$[0 \quad 0 \quad \cdots \quad 0] = P(T)\beta =$$
$$[P_1(T_1)\beta_1 \quad P_2(T_2)\beta_2 \quad \cdots \quad P_n(T_n)\beta_n]$$

$$= [(T_1 - c_1 I_1)q_1(T_1)\beta_1 \quad (T_2 - c_2 I_2)q_2(T_2)\beta_2 \quad \cdots \quad (T_n - c_n I_n)q_n(T_n)\beta_n]$$

$$= [(T_1 - c_1 I_1)\alpha_1 \quad (T_2 - c_2 I_2)\alpha_2 \quad \cdots \quad (T_n - c_n I_n)\alpha_n]$$

$$= (T - cI)\alpha .$$

And this $c$ is a characteristic super value of $T$ .

Conversely suppose that $c$ is a characteristic super value of $T$ .say

$$\mathrm{T}(\alpha) = \mathrm{c}(\alpha)$$
$$= [\mathrm{T}_1(\alpha_1) \quad \mathrm{T}_2(\alpha_2) \quad \cdots \quad \mathrm{T}_n(\alpha_n)]$$
$$= [\mathrm{c}_1\alpha_1 \quad \mathrm{c}_2\alpha_2 \quad \cdots \quad \mathrm{c}_n\alpha_n]$$

With

$$\alpha = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n] \neq [0 \quad 0 \quad \cdots \quad 0]$$

$$P(T)\alpha = [P_1(T_1)\alpha_1 \quad P_2(T_2)\alpha_2 \quad \cdots \quad P_n(T_n)\alpha_n]$$

$$= [P_1(c_1)\alpha_1 \quad P_2(c_2)\alpha_2 \quad \cdots \quad P_n(c_n)\alpha_n]$$

$$= P(c)\alpha$$

But $P(T) = [0 \quad 0 \quad \cdots \quad 0]$ then

$$P(T)\alpha = [0 \quad 0 \quad \cdots \quad 0] = P(c)\alpha$$

So $P(c) = 0$

Theorem(3.2): ( Cayley –Hamilton ) :

If $A$ is a given $n \times n$ matrix and $P(\lambda)$ is the characteristic polynomial of $A$ defined as :

$$P(\lambda) = \det\,(\lambda I_n - A)$$

Then $P(A) = 0$

Proof : Let $B = adj\,(tI_n - A)$

Then according to the right hand fundamental relation of adjugate one has

$$(tI_n - A)\cdot B = \det\,(tI_n - A)I_n = P(t)I_n$$

Since $B$ is also a matrix with polynomials in $t$ as entries , one can for each $i$ collect the coefficients of $t^i$ in each entry to form a matrix $B_i$ of numbers , such that one has

$$B = \sum_{i=0}^{n-1} t^i B_i$$

(The way the entries of $B$ are defined makes clear that no powers higher than $t^{n-1}$ occur). While this looks like a polynomial with matrices as coefficients, we shall not consider such a notion; it is just a way to write a matrix with polynomial entries as linear combination of constant matrices, and the coefficient $t^i$ has been written to the left of the matrix to stress this point of view. Now one can expand the matrix product in our equation by bilinearity :

$$P(t)I_n = (tI_n - A)\cdot B$$

$$= (tI_n - A)\cdot \sum_{i=0}^{n-1} t^i B_i$$

$$= \sum_{i=0}^{n-1} tI_n \cdot t^i B_i - \sum_{i=0}^{n-1} A\cdot t^i B_i$$

$$= \sum_{i=0}^{n-1} t^{i+1} B_i - \sum_{i=0}^{n-1} t^i A\cdot B_i$$

$$= t^n B_{n-1} + \sum_{i=0}^{n-1} t^i (B_{i-1} - A\cdot B_i) - A\cdot B_0$$

Writing

$$P(t)I_n = t^n I_n + t^{n-1}c_{n-1}I + \cdots + tc_1 I_n + c_0 I_n$$

One obtains an equality of two matrices with polynomial entries, written as linear combinations of constant matrices with powers of $t$ as coefficients . Such an equality can hold only if in any matrix position the entry that is multiplied by a given power $t^i$ is the same on both sides; it follows that the constant matrices with coefficient $t^i$ in both expressions must be equal . Writing these equations for $i$ from n down to 0 one finds :

$$B_{n-1} = I_n, \qquad B_{i-1} - A\cdot B_i = c_i I_n$$
$$for\, 0 < i < n, \quad -AB_0 = c_o I_n$$

We multiply the equation of the coefficients of $t^i$ from the left by $A_i$ , and sum up; the left-hand sides form a telescoping sum and cancel completely, which results in the equation :

$$0 = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I_n = P(A)$$

This completes the proof.

Example(3.8) :Consider the super matrix :

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right] = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

Then the super characteristic super polynomial is :

$$P(\lambda) = \left[\lambda^2 - 5\lambda - 2 \mid \lambda^2 - 2\lambda - 1\right]$$

and

$$P(A) = \left[A_1^{\,2} - 5A_1 - 2I \mid A_2^{\,2} - 2A_2 - I\right]$$
$$= \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = [0 \mid 0]$$

## PART THREE

### APPLICATION FOR BVP

We will consider the use of block elimination for the calculation of generalized turning and bifurcation points for two point B.V.P`s . It will be shown that such algorithm will reduce the amount of work required in terms of LU-factorizations to minimal . Since the discretization error of the approximated solution has an asymptotic expansion in terms of even powers of $h$ (the step size ) . This will lead to the use of some type of extrapolation to produce more accurate results . Finite differences will be used to discretize the two point B.V.P`s.

The following examples were used for numerical experimentation .We solved the one-dimensional nonlinear problem :

$$y'' + \lambda e^y = 0 \text{ on the interval } [0,1]$$

With the boundary conditions $y(1) = y(0) = 0$ which has a simple turning point at the critical parameter ; $\lambda = 3.513807$ with the initial guess $\lambda = 3.4$. The results for the fifth iteration with $h = \frac{1}{3}, \frac{1}{6}, \frac{1}{12}, and\ \frac{1}{24}$ using block elimination are given in Table 1.

**Table 1**

| $h$ | $g$ | $\lambda$ |
|---|---|---|
| $\frac{1}{3}$ | 0.143051E-05 | 3.31092 |
| $\frac{1}{6}$ | -0.722452E-05 | 3.46261 |
| $\frac{1}{12}$ | 0.560958E-05 | 3.50110 |
| $\frac{1}{24}$ | 0.471423E-05 | 3.51062 |

Using the results in Table 1 , we applied the Richardson extrapolation and obtained $\lambda = 3.51378$. The results of the application of the extrapolation are given in Table 2 .

**Table 2.**

| $h$ | Number of Extrapolation | | | |
|---|---|---|---|---|
| | 0 | 1 | 2 | 3 |
| $\frac{1}{3}$ | 3.31092 | | | |
| | | 3.51318 | | |
| $\frac{1}{6}$ | 3.46261 | | 3.51398 | |
| | | 3.51393 | | 3.51378 |
| $\frac{1}{12}$ | 3.50110 | | 3.51378 | |
| | | 3.51379 | | |
| $\frac{1}{24}$ | 3.51062 | | | |

It is clear from Table 1 that $g \to 0$ for the various values of $h$ as expected . The CPU time is equal to $0.040$ sec .

Repeating the same calculations for $h = \frac{1}{3}, \frac{1}{6}, \frac{1}{12}, and \frac{1}{24}$ but without using the block-elimination this time . The results were almost the same are given in Table 3 .

**Table 3**

| $h$ | $g$ | $\lambda$ |
|---|---|---|
| $\frac{1}{3}$ | 0.1172142E-03 | 3.31091 |
| $\frac{1}{6}$ | .395983E-05 | 3.46261 |
| $\frac{1}{12}$ | .171911E-05 | 3.50110 |
| $\frac{1}{24}$ | -.404437E-05 | 3.51062 |

Again applying Richardson extrapolation on approximate values of $\lambda_h$ the results are given in Table 4 .

**Table 4.**

| $h$ | Number of Extrapolation | | | |
|---|---|---|---|---|
| | 0 | 1 | 2 | 3 |
| $\frac{1}{3}$ | 3.31092 | | | |
| | | 3.51318 | | |
| $\frac{1}{6}$ | 3.46261 | | 3.51398 | |
| | | 3.51393 | | 3.51378 |
| $\frac{1}{12}$ | 3.50110 | | 3.51378 | |
| | | 3.51379 | | |
| $\frac{1}{24}$ | 3.51062 | | | |

The CPU time is equal to $0.240$ sec . It clear that the CPU time used with block elimination is $\frac{1}{6}$ of the time used without block elimination .

We also solved the system with $h = \frac{1}{50}$ using block elimination . With the same initial guess $\lambda = 3.4$ , we obtain the solution $\lambda = 3.51295$ and $g = 0.98533E - 04$ . The CPU time was $0.190$ sec . This shows that with such a large system , we were not able to obtain the same accuracy as we obtained in Table 3, also the time required is 4 times that needed with Richardson extrapolation .

## REFERENCES


1. E.Rukmangadachari ,Mathematical Methods ,Dorling Kindersley ,India ,2010 .
2. Marshall C. Pease ,Methods of matrix algebra , Stanford Research Institute ,California ,1965 .
3. W. B. Vasantha Kandasamy and Florentin Smarandache ,super linear algebra ,USA ,2008.
4. Basem S.Attili ,Yaqoub Shehadeh ,The use of block elimination for the calculation of some type of singularities efficiently,1992 .
5. http://en.wikipedia.org