*Functions of Bounded Variation and Riemann Stieltjes Integral*

|  |  |  |  |
| --- | --- | --- | --- |
| Ala` Khateeb | Olfat Al-Qawasmi | JumanaHamamde | Khawla Al-Muhtaseb |
| College of Applied Science | College of Applied Science | College of Applied Science | College of Applied Science |
| Palestine Polytechnic University | Palestine Polytechnic University | Palestine Polytechnic University | Palestine Polytechnic University |
| Hebron, Palestine | Hebron, Palestine | Hebron, Palestine | Hebron, Palestine |
| alaa\_lulu\_alaa@hotmail.com | b1l1u1e@hotmail.com | flowers3000@yahoo.com | khawla@ppu.edu |

*Abstract*\_\_ **Functions of bounded variation are a special class of functions with finite variation over an interval. Throughout this seminar, we study the behavior of these functions and give some important theorems to show the essential properties of function of bounded variation. Next, we introduce Riemann Stieltjes integration which is a generalization of the Riemann integral and show the relation between it and functions of bounded variation.**

***Keywords-Bounded; Variation; Riemann; Staieltes Integral .***

1. INTRODUCTION

A function of  bounded variation is a [real](http://en.wikipedia.org/wiki/Real_number)-valued [function](http://en.wikipedia.org/wiki/Function_%28mathematics%29) whose [total variation](http://en.wikipedia.org/wiki/Total_variation) is bounded. In this paper we discuss function of bounded variation and total variation definitions, and illustrative theorems to check whether or not the function is of bounded variation. If the function is of bounded variation, we calculate the total variation of it. Then we introduce the Riemann Stieltjes integral and show its properties. This integral is very important in probability and other science branches.

We divide this paper into three topics, which can be summarized as: we begin by giving some fundamental definitions and theorems needed for our topic. Next we define function of bounded variation and give some important theorems, mainly "Jordan Decomposition Theorem", which shows the close relation between the function of bounded variation and monotonic functions, and that the functions of bounded variation are generated by monotonic functions. Finally, we define Riemann Stieltjes integral which is a generalization of the Riemann integral. Some properties of this integral are discussed, we show the relation between function of bounded variation and Riemann Stieltjes integral. At last we introduce some application of the Riemann Stieltjes integral.

1. PRELIMINERES

Before we define functions of bounded variation, we must lay some fundamental definitions and theorems in order to understand this class of functions.

***1. Partition:*** If is close and bounded interval, a set of points = {, satisfying the inequalities *a=<<…<<=b* is called partition of

We denote to be the set of all partitions of

.

***2. Refinement of* :** A partition of is said to be finer than (or a refinement of ) if .

***3.*** Let : ] be a function. Then is said to be

1. Increasing on if for every ,

.

1. Decreasing on [*a, b*] if for every ,

*x < y*(*x*)(*y*).

1. Monotone if is either increasing or decreasing on .

***4. Bounded Function:*** A functionif there exists a constant such that for all

 is said to be bounded on .

***5. Uniformly Continuous Function:*** Let and let .We say that is uniformly continuous on if for each there exist a such that if

are any numbers satisfying

***6. Mean Value Theorem:*** Suppose that is continuous on a closed interval , and that has a derivative in the open interval .Then there exist at least one point in such that

***7. The Norm of a Partition***  is defined to be the length of the largest subinterval of and it is denoted by , that is, if ={} is partition of then

***8.*** Let be a nonempty subset of :

1. The set is said to be bounded above if there exist a number such that *s* for all

. Each such number is called an upper bound of .

1. The set is said to be bounded below if there exist a number such that for all

. Each such number is called a lower bound of .

1. A set is said to be bounded if it is both bounded above and below.

***9. Supremum:*** If a set is bounded above, then a number is said to be a supremum of if it satisfies the conditions:

1. is an upper bound of .
2. If is any upper bound of , then.

***10. Infimum:*** If a set is bounded below, then a number is said to be an infimum of if it satisfies the conditions:

1. is a lower bound of.
2. ifis any lower bound of , then.

***11. Additive Property Theorem:*** Given a nonempty subsets and of let denote the set

*C*

If each of and has a supremum, then has

a supremum and

1. FUNCTIONS OF BOUNDED VARIATION: DEFINITIONS AND THEOREMS

Function of bounded variation is one of the basic concepts in mathematical analysis, which serves mathematics pure and applied. In this brief chapter we discuss functions of bounded variation and total variation definitions, and illustrative theorems to see whether or not the function is of bounded variation, and if the function is of bounded variation, we calculate the total variation of it.

1. ***Definition of Function of Bounded Variation:***

***Definition 1:*** Let be defined on a closed bounded interval, if is a partition of. Write

(7)

If there exist a positive number such that

where is the set of all partitions of, then is said to be of bounded variation on .

***Example 1:***

Let be a function defined on , and let be a partition of then the variation is given as:

 .

Since is increasing on we have,

 .

We conclude that, for any partition of ,

 .

1. ***Basic Theorems:***

***Theorem 1:*** If is monotonic on, then is of bounded variation on.

***Proof:***

Let be increasing on. Then for every partition of we have,

Hence, is of bounded variation on.

In the same way, we can show that decreasing functions are of bounded variation on.

***Theorem 2:*** If is continues on and differentiable on, such that is bounded, then is of bounded variation on.

***Proof:***

Since is bounded on an open interval, such that

for all in

Let be any partition of, by applying the Mean Value Theorem to on, such that,

=

and take the summation of both sides, we get:

=

 ≤

 =

Hence, is of bounded variation on.

***Remark:***

If is bounded, then dose not necessary be of bounded variation. For example, the function is monotonic, and so is of bounded variation on. However, is not bounded, since

as.



Figure 1. The graph of .



Figure 2.The graph of .

***Theorem 3:*** If is of bounded variation on, then is bounded on.

***Proof:***

Let , consider the partition of , such that

Since is of bounded variation on,

so,

also,

 ≤

which implies,

 ≤

Hence,

(10)

So, is bounded.

***Example 2:***

Show that:

is not of bounded variation on , although is bounded and continuous**.**

Figure3.

Figure 3. The graph of

***Solution:***

Clearly, so is bounded.

implies that is right continues at, so is continues on .

Let us take the partition, the subinterval is . If is even then is odd,

so,

Similarly, if is odd and is even,

but, is divergent as .

Hence, is not of bounded variation on .

***Note that:***

If we take the same example on any closed interval does not contain , say is continuous, exist and bounded, then is of boundedvariation by Theorem 2.

***Remark:***

A function of bounded variation is not necessarily continuous. For example: Let be the greatest integer function less or equal than. Then is of bounded variation on. That is because is increasing, but it is discontinuous.

***Example 3:***

 .

Determine whether or not is of bounded variation on

***Solution:***

is continuous on , on .Therefore.

 exists and bounded on , so is of bounded variation on by ***Theorem 2***.

1. ***Total Variation:***

***Definition 2:*** Let be of bounded variation on, and let denote the sum corresponding to the of. The number defined as follows:

(11)

is called the total variation of on .

***Note:***

A function : is constant if and only if is of bounded variation and .

***Example 4:***

Let be a function defined on , and let be a partition of , then

so,

 .

***Theorem 4:*** Let and be two functions of bounded variation on. Then their sum, difference, and product are functions of bounded variation on, and we have,

1. . (12)
2. . (13)

where

 ,

.

***Proof:***

1. Let , and let be any partition of, we have,

 so,

 .

 Since are each of bounded variation,

 such that,

 and

 so,

 Hence, is of bounded variation.

 Now,

 hence,

 In similar manner we can prove the case

1. Let , and let be any partition of, then

 so,

 sinceand are each of bounded variation we have,

 we conclude that, is of bounded variation on .

 Now,

 Hence,

***Theorem 5:*** Let be function of bounded variation on and assume that is bounded away from , that is suppose a positive number such that,

.Then is also of bounded variation on , and.

***Proof:***

We conclude that, is of bounded variation on , and.

***Corollary:***

Let and be functions of bounded variation on and let be a constant, then

* is of bounded variation on.
* If is of bounded variation on , then is of bounded variation on

***Theorem 6:*** Let be function of bounded variation on and assume that . Then is of bounded variation on and on, and

***Proof:***

Let be a partition of, and let be a partition of then, is a partition of.

If denotes the sum corresponding to any partition. We can write,

so from (16) we have,

i.e.

which means that is of bounded variation on .

and

i.e.

which means is of bounded variation on .

From (16) we can also obtain the inequality,

(17)

To obtain the reverse inequality, let

 and let be the partition obtained by adjoining the point .

If then we have,

and hence,

The corresponding sums for all these partitions are connected by the relation

Therefore, is an upper bound for every sum , since this cannot be smaller than the least upper bound, we must have,

(18)

From (17) + (18) we complete the prove.

***Corollary:***

If is of bounded variation on , then is of bounded variation also on any subinterval of.

1. ***Total Variation as a Function of:***

***Theorem 7:*** Let be of bounded variation on . Let be defined on as follows:

,. Then

1. is an increasing on .
2. is an increasing on .

***Proof:***

1. If are two points in such that , then

Therefore

Hence, is an increasing on .

1. Let , if .

If we have,

 but from the definition of we have,

 this means that,

 Hence, is an increasing on .

***Example 5:***

Let be a function defined on. Then we have the following:

* is increasing on .
* is also increasing on.

Figure 4. The graph of.



Figure 5.The graph of .

***Jordan`s Decomposition Theorem 8:*** A bounded function is of bounded variation, if and only if, there exist two increasing functions and define on such that

***Proof:***

"Only if " part:

Let us define , so is increasing. Let . Then if,

 .

Therefore, whenever .Thus, and are increasing, and .

"If " part :

 Since every bounded monotone function is of bounded variation and the difference of two such function is also of bounded variation, the 'if' part hold.

1. THE RIEMANN – STIELTJES INTEGRAL

In [mathematics](http://en.wikipedia.org/wiki/Mathematics), the Riemann – Stieltjes integral is a generalization of the [Riemann integral](http://en.wikipedia.org/wiki/Riemann_integral), named after [Bernhard Riemann](http://en.wikipedia.org/wiki/Bernhard_Riemann) and [Thomas JoannesStieltjes](http://en.wikipedia.org/wiki/Thomas_Joannes_Stieltjes). The definition of this integral was first published in 1894 by Stieltjes.

1. ***Definition of Riemann – Stieltjes Integral:***

***Definition 3:*** Let be a partition of and let be a point in the subinterval A sum of the form

is called a Riemann- Stieltjes sum of with respect to .

The symbol denotes the difference

, so that

***Definition 4:*** The generalized Riemann–Stieltjes integral of with respect to is a number such that for every , there exists a partition such that for every partition finer than ,

for every choice of points in

When such a number exists, it is uniquely determined and is denoted by

or by

we also say that the Riemann-Stieltjes integral exists, and we write "".

***Note that:***

The functions and are referred to as the integrand and the integrator, respectively.

***Note that:***

In the special case when, we write instead of , and instead of The integral is then called a Riemann integral and is denoted by or by

***Remark:***

The numerical value of depends only on ,, , and , and does not depend on the symbol.

1. ***Monotonically Increasing Integrators***

***(Upper and Lower Riemann– Stieltjes integrals):***

***Definition 5:*** Let be a partition of and let

Then the upper Stieltjes sum is defined as

and the lower Stieltjes sum is ofwith respect tofor the partition .

***Note that:***

i. We always have for all on []. If is increasing on , then for all and we can also write

 that is, for all partition of , then

 ii. If[*k-1 , k*], then

 when is increasing on, we have

, and

These inequalities relate the upper and lower sums to Riemann -Stieltjes sums, and do not necessarily hold when is not an increasing function.

The next theorem shows that, for increasing , the refinement of the partition increases the lower sums and decreases the upper sums.

***Theorem 9:*** Assume that is increasing on :

i. Ifis refinement than, we have

 and

(28)

 ii. For any two partition *1* and *2*, we have

***Definition 6:*** Assume that is an increasing on*.* The upper Stieltjes integral of with respect to is defined as follows:

}.(30)

The lower Stieltjes integral is similarly defined :

}.(31)

Where is the set of all possible partition of.

***Note that:***

We sometimes write (,) and(,) for the upper and lower integrals. In the special case where , the upper and lower sums are denoted by and and are called upper and lower Riemann sums.

***Definition 7:*** A bounded real–valued function is Riemann–Stieltjes integral with respect to on if (,) (,).

***Example 6:***

Let and define on [*,*] as follows :

show that is not Riemann–Stieltjes integral with respect to , where is the set of all rational numbers.

***Solution :***

For every partition of , we have and , since every subinterval contains both rational and irrational numbers. Therefore, and for all . It follows that we have, for,

and

Therefore,

((,

thenis not Riemann–Stieltjes integral.

1. ***Linear Properties:***

***Theorem 10:*** The linear combination of Riemann–Stieltjes integral functions is Riemann–Stieltjes integral, and for any () and if () on , we have

whereand

***Theorem 11:*** If () and if () on, thenon (for any two constant and ) and we have

***Theorem 12:*** Assume thatIfand exist, then also exist and we have

***Definition 8:*** If, we define

whenever exists. We also define
The equation in(***Theorem 12***) can now be written as follows:

 (35)

1. ***Function of Bounded Variation and Riemann – Stieltjes Integral:***

Bounded variation is important to the existence of Riemann– Stieltjes integral. Jordan's Decomposition Theorem plays an important rule in developing the relation between function of bounded variation and Riemann – Stieltjes integral.

***Theorem 13:*** Suppose that is continuous on, and that is of bounded variation on *.* Then the Riemann–Stieltjes integral exists.

***Proof:***

Assume that is an increasing function. Because is of bounded variation, we can write as difference of two increasing functions. i.e (Jordan's Decomposition Theorem), we can write

Let be a partition of*.* Then

}

it is enough to show

},

to prove the Riemann–Stieltjes integral exists.

Now, if is constant on then , thus as do the upper and lower sums.

If is not constant, given, since is continuous on, we conclude that is uniform continuous.Which implies there such that if , then

where,

Therefore, if

Hence,

Thus,

so the Riemann–Stieltjes integral of with respect to exists.

If is decreasing function prove in a similar way.

1. ***Riemann- Stieltjes Integral in Probability Theory:***

In probability theory many concepts are defined using Riemann–Stieltjes integral such that expected value. Here we will define the expected value for a random variable .

***Definition 9:***The sample space is the collection of all possible outcomes that might be observed for random experiment. This set is usually denoted by .

***Definition 10 :*** A random variable is a real valued function on a sample space , that assigns to each elementreal number .

 ***Definition 11:*** The range (or space ) of , say is the set of all possible values of .

***Definition 12 :*** Let be a random variable with range . The cumulative distribution function of the random variable , is defined for all real numbers

by

whereis a probability function .

***Theorem 14:*** Let be a random variable with cumulative distribution function and range ,then

1. then

 ***Definition 13 :*** Let be a random variable with range . The expectation (or expected value) of , denoted by is defined by

Where is continuous random variable.

Now since bounded and increasing function then is Riemann- Stieltjes Integral. So we define using Riemann–Stieltjes integral definition.

***Example7:***

Let *X* be uniform distribution on , , then

 . (38)

##### REFERENCES

1. Apostol, T.M., Mathematical Analysis, 2nd edition, Addison-Wesley, 1974.
2. Bartle, R. and Sherbert, D., Introduction to Real Analysis, 3rd edition,Wiley, 2000.
3. Chatterjee, Dipak, Real analysis, Prentice-Hall of India, New Delhi, India, 2005.
4. Eideh, A.H., Fundamentals of Probability Theory, VDM -Verlag, Germany, 2009.
5. Kolmogorov, A. N. and Fomine,S. V., Introductory RealAnalysis, 1st edition, Dover Publications, New York, 1975.
6. Royden, H. L., Real Analysis, 3rd edition, Collire MacMillan, 1988.