Orthogonal Polynomials Approximation

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Abstract- **In this work we use least square approximation method to approximate continuous function $f(x)$ to polynomial $P_n(x)$ of degree at most n, then we use a different technique based on orthogonal polynomials to facilitate least square approximation polynomial to continuous function. Then we devote to study collections of orthogonal polynomials like Legendre polynomials and Chebyshev polynomials. Then we use Chebyshev polynomials to minimize approximation errors and to reduce the degree of approximated polynomials.**

Keywords-Least squares; linearly Independent; Orthogonal Functions; Chebyshev polynomial; monic polynomial..

1. INTRODUCTION

The study of Approximation Theory involves two general types of problems. One problem arise when a function is given explicitly, but we wish to find a simpler type of function , such polynomial, that can be used to determine approximate values of the given function. The other problem in Approximation theory is concerned in fitting functions to given date and finding the best function in a certain class to represent the data. For example: the Taylor Polynomial of degree n about the number x_0 is an excellent approximation to an $(n+1)$-times differentiable function f in a small neighborhood of x_0. The Lagrange interpolating polynomials can be discussed both as Approximating polynomials and as polynomials to fit cretin data. In this seminar limitations to these techniques are discussed and other ways of approach are considered. we use Least square Approximation method to approximate continuous function $f(x)$ to a polynomial $p_n(x)$ of degree at most n, which minimize the error $\int_a^b [f(x)-p_n(x)]^2 dx$.

Next, we discuss linearly independent and orthogonal functions, to facilitate the discussion of a different technique to obtain a least square Approximation.

Finally, We devote to study collections of orthogonal polynomials like Legendre polynomials ,Which are orthogonal with respect to the weight function $\omega(x)=1$, and Chebyshev polynomials, Which are orthogonal with respect to the weight function $\omega(x)=\left(1-x^2\right)^{-1/2}$. In this seminar we give their definition, and show that they satisfy the required orthogonality properties, Then we use Chebyshev polynomials to get an optimal placing of interpolating points, to minimize the error in Lagrange interpolations, and as a means of reducing the degree of an interpolating polynomial with minimal loss of accuracy.

2. Orthogonal Polynomials and Least Squares Approximation

2.1 *LEAST SQUARES APPROXIMATION*

We will use the Least Squares Approximation method to approximate a continuous function $f(x)$ to a polynomial $P_n(x)$ of degree at most n.

Suppose $f \in C[a,b]$ and that a polynomial P_n of degree at most n is required that will minimize the error

$$\int_a^b [f(x) - P_n(x)]^2 dx .$$

To determine a least square approximation polynomial; that is, a polynomial to minimize this expression, let

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k$$

and define, as shown in Figure 1,

$$E \equiv E_2(a_0, a_1, \ldots, a_n) = \int_a^b \left(f(x) - \sum_{k=0}^{n} a_k x^k \right)^2 dx .$$

We aim to find real coefficients $a_0, a_1, \ldots, a_n$ that will minimize E. A necessary condition for the numbers $a_0, a_1, \ldots, a_n$ to minimize E is that

$$\frac{\partial E}{\partial a_j} = 0, \qquad \text{for each } j = 0, 1, \ldots, n$$

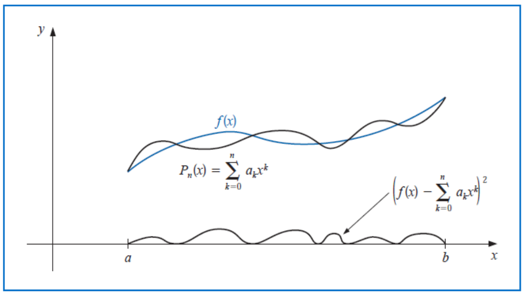

Figure 1

Since

$$E = \int_a^b [f(x)]^2 dx - 2\sum_{k=0}^{n} a_k \int_a^b x^k f(x) dx + \int_a^b \left(\sum_{k=0}^{n} a_k x^k \right)^2 dx$$

we have

$$\frac{\partial E}{\partial a_j} = -2\int_a^b x^j f(x) dx + 2\sum_{k=0}^{n} a_k \int_a^b x^{j+k} dx = 0$$

Hence, to find $P_n(x)$, the $(n+1)$ linear **normal equations**

$$\sum_{k=0}^{n} a_k \int_a^b x^{j+k} dx = \int_a^b x^j f(x) dx, \qquad \text{for each } j = 0, 1, \ldots, n$$

must be solved for the $(n+1)$ unknowns a_j. Since the normal equations always have a unique solution by the following Lemma.

Lemma 1 : The normal equations always have a unique solution $f \in C[a,b]$.

Proof:

The normal equations are

$$\sum_{k=0}^{n} a_k \int_a^b x^{j+k} dx = \int_a^b x^j f(x) dx , \quad \text{for each } j = 0, \ldots, n$$

Let $b_{jk} = \int_a^b x^{j+k} dx$, for each $j = 0, \ldots, n$ and $k = 0, \ldots, n$

and let $B = (b_{jk})$.

Further, let $a = (a_0, \ldots, a_n)^t$ and

$$g = \left(\int_a^b f(x) dx, \ldots, \int_a^b x^n f(x) dx\right)^t .$$

Then the normal equations produce the linear system

$$Ba = g .$$

To show that the normal equation have a unique solution, it suffices to show that if $f = 0$ then $a = 0$.

If $f = 0$, then

$$\sum_{k=0}^{n} a_k \int_a^b x^{j+k} dx = 0 , \quad \text{for each } j = 0, \ldots, n$$

and

$$\sum_{k=0}^{n} a_j a_k \int_a^b x^{j+k} dx = 0 , \quad \text{for each } j = 0, \ldots, n$$

and summing over j gives

$$\sum_{j=0}^{n} \sum_{k=0}^{n} a_j a_k \int_a^b x^{j+k} dx = 0$$

Thus,

$$\int_a^b \sum_{j=0}^{n} \sum_{k=0}^{n} a_j x^j a_k x^k dx = 0$$

and

$$\int_a^b \left(\sum_{j=0}^{n} a_j x^j \right)^2 dx = 0$$

Define $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, then $\int_a^b (P(x))^2 dx = 0$

and $P(x) \equiv 0$.

This implies that $a_0 = a_1 = \cdots = a_n$, so that $a = 0$.

Hence , the matrix B is nonsingular, and the normal equation have a unique solution.

Example 2.1.1 :

Find the least squares approximation polynomial of degree 2 for the function $f(x) = \sin \pi x$ on the interval $[0,1]$.

Solution:

The normal equation for $P_2(x) = a_2 x^2 + a_1 x + a_0$ are when $j = 0$

$$\sum_{k=0}^{2} a_k \int_0^1 x^{j+k} dx = a_0 \int_0^1 1 dx + a_1 \int_0^1 x dx + a_2 \int_0^1 x^2 dx = \int_0^1 \sin \pi x dx$$

When $j=1$

$$\sum_{k=0}^{2} a_k \int_0^1 x^{j+k}dx = a_0\int_0^1 x dx + a_1\int_0^1 x^2 dx + a_2\int_0^1 x^3 dx = \int_0^1 x\sin\pi x dx$$

When $j=2$

$$\sum_{k=0}^{2} a_k \int_0^1 x^{j+k}dx = a_0\int_0^1 x^2 dx + a_1\int_0^1 x^3 dx + a_2\int_0^1 x^4 dx = \int_0^1 x^2\sin\pi x dx$$

Performing the integration yields

$$a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 = \frac{2}{\pi},9$$

$$\frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 = \frac{1}{\pi},$$

$$\frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2 = \frac{\pi^2-4}{\pi^3}.$$

These there equations in there unknowns can be solved to obtain

$$a_0 = \frac{12\pi^2-120}{\pi^3} \approx -0.050465$$

and $$a_1 = -a_2 = \frac{720-60\pi^2}{\pi^3} \approx 4.12251.$$

The least squares polynomial approximation of degree 2 for $f(x)=\sin\pi x$ on [0,1] is

$P_2 = -4.12251x^2 + 4.12251x - 0.050465.$ (See Figure 2)

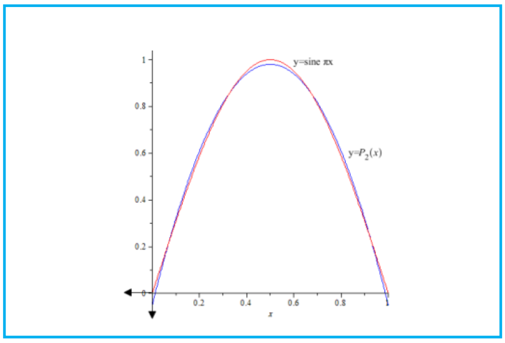

Figure 2

Example 2.1.1 illustrates a difficulty in obtaining a least squares polynomial approximation.

An $(n+1)\times(n+1)$ linear system for unknowns $a_0,\dots,a_n$ must be solved, and the coefficients in the linear system of the form

$$\int_a^b x^{j+k}dx = \frac{b^{j+k+1}-a^{j+k+1}}{j+k+1},$$

A linear system that does not have an easily computed numerical solution. So, we will use a different technique to obtain Least Square Approximation, and we will introduce some new concepts in the following two sections like (Linearly independence and Orthogonality) to facilitate the discussion.

2.2 *Linearly Independent Functions*

Definition 2.2.1 : The set of function $\{\phi_0,\dots,\phi_n\}$ is said to be **Linearly independent** on [a, b] if , whenever

$$c_0\phi_0 + c_1\phi_1 + \cdots + c_n\phi_n = 0, \quad \text{for all } x\in[a,b],$$

We have $c_0 = c_1 = \cdots = c_n = 0.$ Otherwise the set of function is said to be **Linearly dependent**.

Theorem 2.2.1 : suppose that, for each $j = 0, \dots, n$, $\phi_j(x)$ is a polynomial of degree j. Then $\{\phi_0,\dots,\phi_n\}$ is linearly independent on any interval [a, b].

Proof:

Let $c_0,\dots,c_n$ be real numbers for which

$$\mathrm{P}(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) = 0, \text{ for all } x\in[a,b],$$

The polynomial $P(x)$ vanishes on [a, b], so it must be the zero polynomial, and the coefficients of all the powers of x are zero. In particular, the coefficient of x^n is zeros. But $c_n\phi_n(x)$ is the only term in $P(x)$ that contains x^n, so we must have $c_n = 0.$ Hence

$$P(x) = \sum_{j=0}^{n-1} c_j\phi_j(x).$$

In this representation of $P(x)$, the only term contains a power of x^{n-1} is $c_{n-1}\phi_{n-1}(x)$, so this term must also be zero and

$$P(x) = \sum_{j=0}^{n-2} c_j\phi_j(x).$$

In like manner, the remaining constants $c_{n-2}, c_{n-3},\dots,c_1,c_0$ are all zero, which implies that $\{\phi_0,\phi_1,\dots,\phi_n\}$ is linearly independent on [a, b].

Example 2.2.1:

Let $\phi_0(x)=2, \phi_1(x)=x-3,$ and $\phi_2(x)=x^2+2x+7,$ $Q(x)=a_0+a_1x+a_2x^2.$ Show that there exist c_0, c_1 and c_2 such that $Q(x)=c_0\phi_0(x)+c_1\phi_1(x)+c_2\phi_2(x).$

Solution:

By Theorem 2.2.1, $\{\phi_0,\phi_1,\phi_2\}$ is linearly independent on any interval [a, b].

First note that

$$1=\frac{1}{2}\phi_0(x), \qquad x=\phi_1(x)+3=\phi_1(x)+\frac{3}{2}\phi_0(x)$$

and

$$x^2=\phi_2(x)-2x-7=\phi_2(x)-2\left[\phi_1(x)=\frac{3}{2}\phi_0(x)\right]-7\left[\frac{1}{2}\phi_0(x)\right]$$
$$=\phi_2(x)-2\phi_1(x)-\frac{13}{2}\phi_0(x).$$

Hence

$$Q(x)=a_0\left[\frac{1}{2}\phi_0(x)\right]+a_1\left[\phi_1(x)+\frac{3}{2}\phi_0\right]+a_2\left[\phi_2(x)-2\phi_1+\frac{13}{2}\phi_0\right]$$
$$=\left(\frac{1}{2}a_0+\frac{3}{2}a_1-\frac{13}{2}a_2\right)\phi_0(x)+\left(a_1-2a_2\right)\phi_1(x)+a_2\phi_2(x).$$

Theorem 2.2.2 : suppose that $\{\phi_0(x),\phi_1(x),...,\phi_n(x)\}$ is a collection of linearly independent polynomials in $\prod_n$. Then any polynomial in $\prod_n$ can be written uniquely as a linear combination of $\phi_0(x),\phi_1(x),...,\phi_n(x)$.

Proof:

let $\{\phi_0(x),\phi_1(x),...,\phi_n(x)\}$ be a linearly independent set of polynomial in $\prod_n$.

For each $i=0,1,...,n$, let $\phi_i(x)=\sum_{k=0}^{n} b_{ki}x^k$,

let $Q(x)=\sum_{k=0}^{n} a_k x^k \in \prod_n$.

We want to find constant $c_0,...,c_n$ so that

$$Q(x)=\sum_{i=0}^{n} c_i\phi_i(x)$$

This equation becomes

$$\sum_{k=0}^{n} a_k x^k = \sum_{i=0}^{n} c_i (\sum_{k=0}^{n} b_{ki}x^k)$$

and

$$\sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} (\sum_{i=0}^{n} c_i b_{ki})x^k$$

$$\sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} (\sum_{i=0}^{n} b_{ki} c_i)x^k$$

But $\{1, x, ..., x^n\}$ is linearly independent, so, for each $k=0,...,n$ we have

$$\sum_{i=0}^{n} b_{ki}c_i = a_k$$

Which expands to the linear system

$$\begin{bmatrix} b_{01} & b_{02}\ldots & b_{0n} \\ b_{11} & b_{12}\ldots & b_{1n} \\ \vdots & \vdots & \vdots \\ b_{n1} & b_{n2}\ldots & b_{nn} \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}=\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

This linear system must have a unique solution $\{c_0, c_1, ..., c_n\}$. Or else there is a nontrivial set of constant $\{c'_0, c'_1, ..., c'_n\}$, or which

$$\begin{bmatrix} b_{01} & b_{02}\ldots & b_{0n} \\ b_{11} & b_{12}\ldots & b_{1n} \\ \vdots & \vdots & \vdots \\ b_{n1} & b_{n2}\ldots & b_{nn} \end{bmatrix}\begin{bmatrix} c'_0 \\ c'_1 \\ \vdots \\ c'_n \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Thus

$$c'_0\phi_0(x)+c'_1\phi_1(x)+...+c'_n\phi_n(x)=\sum_{k=0}^{n} 0x^k = 0$$

Which contradicts the linear independences of the set $\{\phi_0,\phi_1,...,\phi_n\}$.

Thus, there is a unique set of constant $\{c_0, c_1, ..., c_n\}$, for which

$$Q(x)=c_0\phi_0(x)+c_1\phi_1(x)+...+c_n\phi_n(x)$$

2.3 *Orthogonal Functions*

Definition 2.3.1 : An integrable function ω is called a **weight function** on the interval I if $\omega(x) \geq 0$, for all x in I, but $\omega(x) \neq 0$ on any subinterval of I.

Note: The purpose of a weight function is to assign varying degrees of importance to approximations on certain portions of the interval. For example, the weight function

$$\omega(x) = \frac{1}{\sqrt{1-x^2}}$$

places less emphasis near the center of the interval $(-1,1)$ and more emphasis when $|x|$ is near 1.

Definition 2.3.2 : Suppose $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is said to be an **orthogonal set of functions** for the interval $[a,b]$ with respect to the weight function ω if

$$\int_a^b \omega(x)\phi_k(x)\phi_j(x)dx = \begin{cases} 0 & ,\text{if } j \neq k \\ \alpha_j > 0 & ,\text{if } j = k. \end{cases}$$

If $\alpha_j = 1$ for each $j = 0,1,\ldots,n$, the set is to be **orthonormal.**

Theorem 2.3.1 : If $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is an orthogonal set of functions on an interval $[a,b]$ with respect to the weight function ω, then the least squares approximation to f on $[a,b]$ with respect to ω is

$$p(x) = \sum_{j=0}^{n} a_j \phi_j(x),$$

where , for each $j = 0,1,\ldots,n$,

$$a_j = \frac{\int_a^b \omega(x)\phi_j(x)f(x)dx}{\int_a^b \omega(x)\left[\phi_j(x)\right]^2 dx} = \frac{1}{\alpha_j}\int_a^b \omega(x)\phi_j(x)f(x)dx.$$

Proof:

suppose $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is a set of linearly independent functions on $[a,b]$ and ω is a weight function for $[a,b]$. Given $f \in C[a,b]$, we seek a linear combination

$$P(x) = \sum_{k=0}^{n} a_k \phi_k(x)$$

to minimize the error

$$E = E(a_0, \ldots, a_n) = \int_a^b \omega(x)\left[f(x) - \sum_{k=0}^{n} a_k \phi_k\right]^2 dx.$$

The normal equations are derived from the fact that for each $j = 0,1,\ldots,n$,

$$0 = \frac{\partial E}{\partial a_j} = 2\int_a^b \omega(x)\left[f(x) - \sum_{k=0}^{n} a_k \phi_k(x)\right]\phi_j(x)dx.$$

$$0 = \int_a^b \left[2\omega(x)\phi_j(x)f(x) - 2\omega(x)\phi_j(x)\sum_{k=0}^{n} a_k \phi_k(x)\right]dx.$$

The system of normal equations can be written

$$\int_a^b \omega(x)f(x)\phi_j(x)dx = \sum_{k=0}^{n} a_k \int_a^b \omega(x)\phi_k(x)\phi_j(x)dx, \text{ for } j = 0,1,\ldots,n.$$

If the functions $\phi_0, \phi_1, \ldots, \phi_n$ can be chosen so that

$$\int_a^b \omega(x)\phi_k(x)\phi_j(x)dx = \begin{cases} 0 & ,\text{if } j \neq k, \\ \alpha_j > 0 & ,\text{if } j = k. \end{cases}$$

then the normal equations will reduce to

$$\int_a^b \omega(x)f(x)\phi_j(x)dx = a_j \int_a^b \omega(x)\left[\phi_j(x)\right]^2 dx = a_j \alpha_j,$$

for each $j = 0,1,\ldots,n$. These are easily solved to give

$$a_j = \frac{1}{\alpha_j}\int_a^b \omega(x)f(x)\phi_j(x)dx.$$

Hence the least squares approximation problem is greatly simplified when the functions $\phi_0, \phi_1, \ldots, \phi_n$ are chosen to satisfy the *Orthogonality* condition .

The next Theorem, which is based on the **Gram-Schmidt process**, describes how to construct orthogonal polynomials on $[a,b]$ with respect to a weight function ω.

Theorem 2.3.2 : The set of polynomial functions $\{\phi_0, \phi_1, \ldots, \phi_n\}$ defined in the following way is orthogonal on $[a,b]$ with respect to the weight function ω.

$\phi_0 = 1, \quad \phi_1 = x - B_1,$ for each x in $[a,b]$,

where

$$B_1 = \frac{\int_a^b x\,\omega(x)\left[\phi_0(x)\right]^2 dx}{\int_a^b \omega(x)\left[\phi_0(x)\right]^2 dx},$$

and when $k \ge 2$,

$\phi_k = (x - B_k)\phi_{k-1}(x) - c_k\,\phi_{k-2}(x)$, for each x in $[a,b]$,

where

$$B_k = \frac{\int_a^b x\,\omega(x)\left[\phi_{k-1}(x)\right]^2 dx}{\int_a^b \omega(x)\left[\phi_{k-1}(x)\right]^2 dx},$$

and

$$c_k = \frac{\int_a^b x\,\omega(x)\phi_{k-1}\phi_{k-2}dx}{\int_a^b \omega(x)\left[\phi_{k-2}(x)\right]^2 dx}.$$

Corollary 2.3.1 : For any the set of polynomial functions $\{\phi_0,\phi_1,\ldots,\phi_n\}$ given in Theorem 1.3.2 is linearly independent on $[a,b]$ and

$$\int_a^b \omega(x)\phi_n(x)Q_k(x)dx = 0,$$

for any polynomial $Q_k(x)$ of degree $k < n$.

Proof:

For each $k = 0,1,\ldots,n$, $\phi_k(x)$ is polynomial of degree k. So Theorem 2.2.1 implies that $\{\phi_0,\ldots,\phi_n\}$ is a linearly independent set.

Let $Q_k(x)$ be a polynomial of degree $k < n$. By Theorem 1.2.2 there exist numbers $c_0,\ldots,c_k$ such that

$$Q_k(x) = \sum_{j=0}^{k} c_j\phi_j(x).$$

Because ϕ_n is orthogonal to ϕ_j for each $j = 0,1,\ldots,k$ we have

$$\int_a^b \omega(x)\phi_n(x)Q_k(x)dx = \sum_{j=0}^{k} c_j \int_a^b \omega(x)\phi_n(x)\phi_j(x)dx$$
$$= \sum_{j=0}^{k} c_j .0 = 0.$$

As an application to the previous Theorem 2.3.2 we will construct orthogonal Legendre polynomials.

Example 2.3.1:

The set of Legendre polynomials $\{P_n(x)\}$. is orthogonal on $[-1,1]$ with respect to the weight function $\omega(x) \equiv 1$. The classical definition of the Legendre polynomials requires that $P_n(1) = 1$ for each n, and a recursive relation is used to generate the polynomials when $n \ge 2$. This normalization will not be needed in our discussion, and the least squares approximating polynomials generated in either case are essentially the same.

Using the Gram-Schmidt process with $P_0(x) \equiv 1$ gives

$$B_1 = \frac{\int_{-1}^{1} x dx}{\int_{-1}^{1} dx} = 0 \quad \text{and} \quad P_1(x) = (x - B_1)P_0 = x.$$

Also,

$$B_2 = \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1} x^2 dx} = 0 \quad \text{and} \quad C_2 = \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} 1dx} = \frac{1}{3},$$

So,

$$P_2(x) = (x - B_2)P_1(x) - C_2P_0(x) = (x-0)x - \frac{1}{3}.1$$
$$= x^2 - \frac{1}{3}.$$

The higher-degree Legendre polynomials shown in Figure 4 are derived in the same manner.

Thus

$$P_3(x) = xP_2(x) - \frac{4}{5}P_1(x) = x^3 - \frac{1}{3}x - \frac{4}{15}x = x^3 - \frac{3}{5}x.$$

The next two Legendre polynomials are

$$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35},$$

and

$$P_5(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x.$$

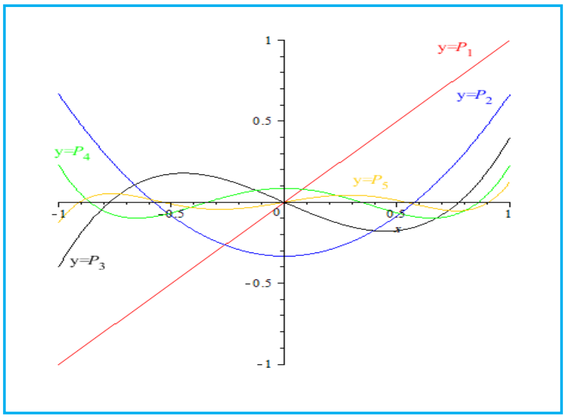

Figure 4

3. Chebyshev polynomials and Economization of Power Series

3.1 Chebyshev polynomials

Definition 3.1.1 : Chebyshev polynomial of degree $n \ge 0$ is defined as

$$T_n(x) = \cos(n\cos^{-1}x), \quad \text{for each} \quad x \in [-1,1].$$

or, in a more instructive form,

$$T_n(x) = \cos(n\theta), \quad \text{since } x = \cos\theta, \quad \theta \in [0,\pi].$$

The Properties of Chebyshev Polynomials

i. **Range:** $T_n(x) \in [-1,1]$ For odd n, $T_n(-1) = -1$ and $T_n(1) = 1$. For even n, $T_n(-1) = 1$ and $T_n(1) = 1$.

ii. **Symmetry:** For odd n, $T_n(x)$ is an odd function, *i.e.*, $T_n(-x) = -T_n(x)$. For even n, $T_n(x)$ is an even function, *i.e.*, $T_n(-x) = T_n(x)$.

iii. **Roots:** The Chebyshev polynomial $T_n(x)$ of degree $n \ge 1$ has n simple zeros in $[-1,1]$ at

$$\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad \text{for } k = 1,2,\dots,n.$$

Proof:

Let $\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right)$, for $k = 1,2,\dots,n$.

Then

$$T_n(\bar{x}_k) = \cos(n\cos^{-1}\bar{x}_k) = \cos\left(n\cos^{-1}\left(\cos\left(\frac{2k-1}{2n}\pi\right)\right)\right)$$

$$= \cos\left(\frac{2k-1}{2}\pi\right) = 0.$$

But the $\bar{x}_k$ are distinct and $T_n(x)$ is a polynomial of degree n, so all the zeros of $T_n(x)$ must have this form.

iv. **Extrema:** $T_n(x)$ has $n+1$ distinct extrema in $[-1,1]$. All extrema give the value of either -1 or 1. The extrema have the following expression:

$$\bar{x}'_k = \cos\left(\frac{k\pi}{n}\right) \quad \text{with} \quad T_n(\bar{x}'_k) = (-1)^k, \text{ for each } k = 0,1,\dots,n.$$

Proof:

note that

$$T'_n(x) = \frac{d}{dx}[\cos(n\cos^{-1}x)] = \frac{n\sin(n\cos^{-1}x)}{\sqrt{1-x^2}},$$

And that, when $k = 1,2,\dots,n-1$,

$$T_n(\bar{x}'_k) = \frac{n\sin\left(n\cos^{-1}\left(\cos\left(\frac{k\pi}{n}\right)\right)\right)}{\sqrt{1-\left[\cos\left(\frac{k\pi}{n}\right)\right]^2}} = \frac{n\sin(k\pi)}{\sin\left(\frac{k\pi}{n}\right)} = 0.$$

Since $T_n(x)$ is a polynomial of degree n, its derivative $T'_n(x)$ is a polynomial of degree $(n-1)$, and all the zeros of $T'_n(x)$ occur at these $n-1$ distinct points.

The only other possibilities for extrema of $T_n(x)$ occur at the endpoints of the interval $[-1,1]$; that is, at $\bar{x}'_0 = 1$ and at $\bar{x}'_n = -1$.

For any $k = 0,1,\dots,n$ we have

$$T_n(\bar{x}'_k) = \cos\left(n\cos^{-1}\left(\cos\left(\frac{k\pi}{n}\right)\right)\right) = \cos(k\pi) = (-1)^k.$$

So a maximum occurs at each even value of k and a minimum at each odd value.

v. **Orthogonality:** The family of Chebyshev polynomials is orthogonal with the following Chebyshev weighting function:

$$\omega(x) = \frac{1}{\sqrt{1-x^2}}$$

Orthogonality implies that, for two polynomials of the family $T_n(x)$ and $T_m(x)\,(n \neq m)$:

$$\int_{-1}^{1} T_n(x)T_m(x)\omega(x)dx = 0$$

In addition, in case $m = n = 0$

$$\int_{-1}^{1} T_n(x)T_m(x)\omega(x)dx = \pi$$

In case $m = n \neq 0$

$$\int_{-1}^{1} T_n(x)T_m(x)\omega(x)dx = \frac{\pi}{2}$$

Proof:

consider

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \int_{-1}^{1} \frac{\cos(n\cos^{-1}x)\cos(m\cos^{-1}x)}{\sqrt{1-x^2}}dx.$$

Reintroduction the substitution $\theta = \cos^{-1}x$ gives

$$d\theta = -\frac{1}{\sqrt{1-x^2}}dx$$

and

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = -\int_{\pi}^{0} \cos(n\theta)\cos(m\theta)d\theta$$
$$= \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)d\theta.$$

Suppose $\mathrm{n} \neq \mathrm{m}$: Since

$$\cos(n\theta)\cos(m\theta) = \frac{1}{2}\left[\cos(n+m)\theta + \cos(\mathrm{n}-m)\theta\right],$$

We have

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \frac{1}{2}\int_{0}^{\pi} \cos((n+m)\theta)d\theta + \frac{1}{2}\int_{0}^{\pi} \cos((\mathrm{n}-m)\theta)d\theta.$$

$$= \left[\frac{1}{2(n+m)}\sin((n+m)\theta) + \frac{1}{2(n-m)}\sin((n-m)\theta)\right]_0^{\pi} = 0.$$

Suppose $m = n \neq 0$: We have

$$\int_{-1}^{1} \frac{T_n(x)T_n(x)}{\sqrt{1-x^2}}dx = \int_{-1}^{1} \frac{T_n^{\,2}(x)}{\sqrt{1-x^2}}dx$$
$$= \int_{-1}^{1} \frac{\cos(n\cos^{-1}x)\cos(n\cos^{-1}x)}{\sqrt{1-x^2}}dx$$
$$= \int_{-1}^{1} \frac{\cos^2(n\cos^{-1}x)}{\sqrt{1-x^2}}dx.$$

Then change a variable $x = \cos\theta$ produces

$$= \int_{0}^{\pi} (\cos(n\theta))^2 d\theta = \frac{\pi}{2} \quad \text{for each } n \geq 1.$$

Lemma 2: If T_n is a Chebyshev polynomial of degree $n,\ n \geq 1.$ then

$$T_{n+1}(\mathrm{x}) = 2xT_n(\mathrm{x}) - T_{n-1}(\mathrm{x}).$$

Proof:

first note that

$$T_0(x) = \cos 0 = 1 \quad \text{and} \quad T_1(x) = \cos(\cos^{-1}x) = x.$$

For $\mathrm{n} \geq 1$, we introduce the substitution $\theta = \cos^{-1}x$ to change this equation to

$$T_n(\theta(x)) = T_n(\theta) = \cos(\mathrm{n}\theta), \quad \text{where } \theta \in [0, \pi].$$

A recurrence relation is derived noting that

$$T_{n+1}(\theta) = \cos((\mathrm{n}+1)\theta) = \cos\theta\cos(\mathrm{n}\theta) - \sin\theta\sin(\mathrm{n}\theta)$$

and

$$T_{n-1}(\theta) = \cos((\mathrm{n}-1)\theta) = \cos\theta\cos(\mathrm{n}\theta) + \sin\theta\sin(\mathrm{n}\theta)$$

Adding these equation gives

$$T_{n+1}(\theta) = 2\cos\theta\cos(\mathrm{n}\theta) - T_{n-1}(\theta)$$

Returning to the variable $\mathrm{x} = \cos\theta$, we have, for $\mathrm{n} \geq 1$,

$$T_{n+1}(x) = 2x\cos(n\cos^{-1}x) - T_{n-1}(x),$$

That is,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Example 3.1.1

Find $T_2(x)$, $T_3(x)$ & $T_4(x)$ when you know $T_0(x)=1$ and $T_1(x)=x$

Solution:

Because $T_0(x)=\cos 0 = 1$ and $T_1(x)=x$, the recurrence relation implies that the next three Chebyshev polynomials are

$$T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1,$$
$$T_3(x) = 2xT_2(x) - T_1(x) = 4x^3 - 3x,$$

and

$$T_4(x) = 2xT_3(x) - T_2(x) = 8x^4 - 8x^2 + 1.$$

The recurrence relation also implies that when $n \ge 1$, $T_n(x)$ is a polynomial of degree n with leading coefficient 2^{n-1}. The graphs of T_1, T_2, T_3, and T_4 are shown in Figure 5.

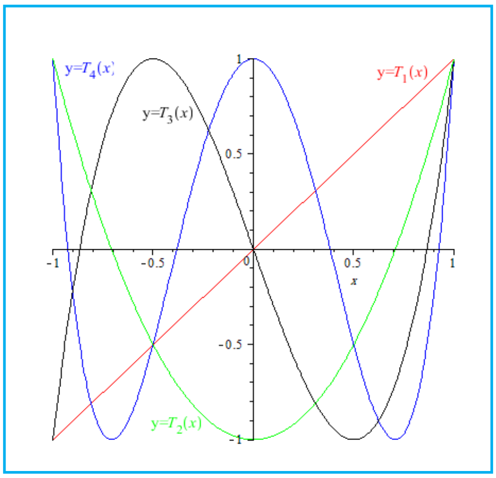

Figure 5

Definition 3.1.2: A monic polynomial is a polynomial with leading coefficient 1.

The monic Chebyshev polynomial $\tilde{T}_n(x)$ are derived from the Chebyshev polynomial $T_n(x)$ by dividing by the leading coefficient 2^{n-1}.

Hence

$$\tilde{T}_0(x) = 1 \quad \text{and} \quad \tilde{T}_n(x) = \frac{1}{2^{n-1}}T_n(x), \quad \text{for each } n \ge 1.$$

The recurrence relationship satisfied by the Chebyshev polynomials implies that

$$\tilde{T}_2(x) = x\tilde{T}_1(x) - \frac{1}{2}\tilde{T}_0(x) \quad \text{and}$$

$$\tilde{T}_{n+1}(x) = x\tilde{T}_n(x) - \frac{1}{4}\tilde{T}_{n-1}(x), \quad \text{for each } n \ge 2.$$

The graphs of $\tilde{T}_1, \tilde{T}_2, \tilde{T}_3, \tilde{T}_4$, and $\tilde{T}_5$ are shown in Figure6.

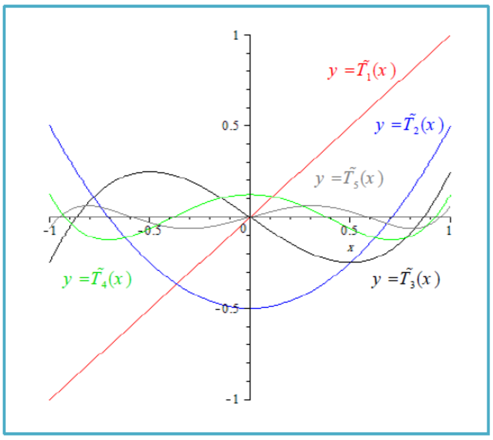

Figure 6

Because $\tilde{T}_n(x)$ is just a multiple of $T_n(x)$, the zeros of $\tilde{T}_n(x)$ also occur at

$$\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad \text{for each } k = 1, 2, \ldots, n,$$

and the extreme values of $\tilde{T}_n(x)$, for $n \ge 1$, occur at

$$\bar{x}'_k = \cos\left(\frac{k\pi}{n}\right), \text{ with } \tilde{T}_n(\bar{x}'_k) = \frac{(-1)^k}{2^{n-1}}, \text{ for each}$$
$$k = 0, 1, \ldots, n. \qquad (1)$$

Definition 3.1.3 : Let $\tilde{\Pi}_n$ denoted the set of all monic polynomials of degree n.

The relation expressed in Eq.(1) leads an important minimization property that distinguishes $\tilde{T}_n(x)$ from the other members of $\tilde{\Pi}_n$

Theorem 3.1.1 : The polynomials of the form $\tilde{T}_n(x)$, when $n \geq 1$, have the property that

$$\frac{1}{2^{n-1}} = \max_{x\in[-1,1]} |\tilde{T}_n(x)| \leq \max_{x\in[-1,1]} |P_n(x)|, \text{ for all } P_n(x) \in \tilde{\Pi}_n.$$

Moreover, equality occurs only if $P_n \equiv \tilde{T}_n$.

Proof:

suppose that $P_n(x) \in \tilde{\Pi}_n$ and that

$$\max_{x\in[-1,1]} |P_n(x)| \leq \frac{1}{2^{n-1}} = \max_{x\in[-1,1]} |\tilde{T}_n(x)|.$$

Let $Q = \tilde{T}_n - P_n$. Then $\tilde{T}_n(x)$ and $P_n(x)$ are both monic polynomials of degree n, so $Q(x)$ is a polynomial of degree at most $(n-1)$. Moreover, at the $n+1$ extreme points $\bar{x}'_k$ of $\tilde{T}_n(x)$, we have

$$Q(\bar{x}'_k) = \tilde{T}_n(\bar{x}'_k) - P_n(\bar{x}'_k) = \frac{(-1)^k}{2^{n-1}} - P_n(\bar{x}'_k).$$

However

$$|P_n(\bar{x}'_k)| \leq \frac{1}{2^{n-1}}, \qquad \text{for each } k = 0,1,...,n,$$

so we have

$Q(\bar{x}'_k) \leq 0$, when k is odd and $Q(\bar{x}'_k) \geq 0$, when k is even.

Since Q is continuous, the Intermediate Value Theorem implies that for each $j = 0,1,...,n-1$ the polynomial $Q(x)$ has at least one zero between $\bar{x}'_j$ and $\bar{x}'_{j+1}$. Thus, Q has at least n zeros in the interval $[-1,1]$. But the degree of $Q(x)$ is less than n, so $Q \equiv 0$. This implies $P_n \equiv \tilde{T}_n$.

3.2 Minimizing Approximation Error

I. ***Minimizing Lagrange Interpolating Error***

Theorem 3.2.1 (The nth Lagrange Interpolating Polynomial): If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct numbers and f is a function whose values are given at these numbers, then a unique polynomial $P(x)$ of degree at most n exists with $f(x_k) = P(x_k)$ for each $k = 0,1,\ldots,n$.

This polynomial is given by

$$P(x) = f(x_0)L_{n,0}(x) + \ldots. + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

where, for each $k = 0,1,\ldots,n$

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}$$
$$= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}$$

Theorem 3.1.1 can be used to answer the question of where to place interpolating nodes to minimize the error in Lagrange interpolation.

The Lagrange interpolating polynomial is the <u>polynomial</u> $P(x)$ of degree $\leq (n-1)$ that passes through n points $(x_1, y_1 = f(x_1))$, $(x_2, y_2 = f(x_2))$, …, $(x_n, y_n = f(x_n))$, and is given by

$$P(x) = \sum_{j=1}^{n} P_j(x),$$

where

$$P_j(x) = y_j \prod_{\substack{k=1 \\ k\neq j}}^{n} \frac{x-x_k}{x_j-x_k}$$

Theorem 3.2.3: If $x_1, x_2, \ldots, x_n$ are distinct numbers in the interval $[a,b]$ and $f \in C^{n+1}[a,b]$ then, for each x in $[a,b]$, a number $\xi(x)$ in (a,b) exist with :

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)...(x-x_n),$$

where $P(x)$ is the Lagrange interpolating polynomial.

Applied to the interval $[-1,1]$ states that, if $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[-1,1]$ and if $f \in C^{n+1}[-1,1]$, then, for each $x \in [-1,1]$, a number $\xi(x)$ exists in $(-1,1)$ with

$$f(x) - P(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)....(x-x_n).$$

Generally, there is no control over $\xi(x)$, so to minimize the error by shrewd placement of the nodes $x_0, x_1, \ldots, x_n$ we choose $x_0, x_1, \ldots, x_n$ to minimize the quantity

$$|(x-x_0)(x-x_1)....(x-x_n)|$$

throughout the interval $[-1,1]$.

Since $(x-x_0)(x-x_1)....(x-x_n)$ is a monic polynomial of degree $(n+1)$, we have just seen that the minimum is obtained when

$$(\mathrm{x}-\mathrm{x}_0)(\mathrm{x}-\mathrm{x}_1)....(\mathrm{x}-\mathrm{x}_\mathrm{n}) = \tilde{T}_{n+1}(x)$$

The maximum value of $|(x-x_0)(x-x_1)....(x-x_n)|$ is smallest when x_k is choose for each $k = 0,1,\ldots.,n$ to be the $(k+1)$st zero of $\tilde{T}_{n+1}(x)$.

Hence we choose x_k to be

$$\bar{x}_{k+1} = \cos\left(\frac{2k+1}{2(n+1)}\pi\right)$$

Because $\max_{x\in[-1,1]} |\tilde{T}_{(n+1)}| = 2^{-n}$, this also implies that

$$\frac{1}{2^{\mathrm{n}}} = \max_{x\in[-1,1]} |(x-\bar{x}_1)\ldots(x-\bar{x}_{n+1})| \le \max_{x\in[-1,1]} |(x-x_0)\ldots(x-x_n)|,$$

for any choice of $x_0, x_1, ..., x_n$ in interval $[-1,1]$. The next corollary follows form these observation.

Corollary 3.2.1: Suppose that $P(x)$ is the interpolating polynomial of degree at most n with nodes at the zeros of $T_{n+1}(x)$. Then

$$\max_{x\in[-1,1]} |f(x) - P(x)| \le \frac{1}{2^n (n+1)!} \max_{x\in[-1,1]} |f^{(n+1)}(x)|, \quad \text{for each } f \in C^{n+1}[-1,1].$$

II. Minimizing Approximation Error on Arbitrary Intervals

The technique for choosing points to minimize the interpolating error is extended to a general closed interval $[a,b]$ by using the change of variables

$$\tilde{x} = \frac{1}{2}[(b-a)x + a + b]$$

to transform the numbers $\tilde{x}_k$ in the interval $[-1,1]$ into the corresponding $\tilde{x}_k$ in the interval $[a,b]$, as shown in the next example.

Example 3.2.1 :

Let $f(x) = xe^x$ on $[0,1.5]$. Compare the values given by the Lagrange polynomial with four equally-spaced nodes with those given by the Lagrange polynomial with nodes given by zeros of the fourth Chebyshev polynomial.

Solution :

The equally-spaced nodes $x_0 = 0$, $x_1 = 0.5$ $x_2 = 1$, and $x_3 = 1.5$

$$L_0(x) = -1.3333x^3 + 4.0000x^2 - 3.6667x + 1,$$
$$L_1(x) = 4.0000x^3 - 10.000x^2 + 6.0000x,$$
$$L_2(x) = -4.0000x^3 + 8.000x^2 - 3.0000x,$$
$$L_3(x) = 1.3333x^3 - 2.000x^2 + 0.66667x,$$

which produces the polynomial

$$P_3(x) = L_0(x)(0) + L_1(x)(0.5e^{0.5}) + L_2(x)e + L_3(x)(1.5e^{1.5})$$
$$= 1.3875x^3 + 0.057570x^2 + 1.2730x.$$

For the second interpolating polynomial, we shift the zeros $\bar{x}_k = \cos\left(\frac{2k+1}{8}\right)\pi$, for $k = 0,1,2,3$, of $\tilde{T}_4$ from $[-1,1]$ to $[0,1.5]$, using the linear transformation.

$$\tilde{x}_k = \frac{1}{2}\left[(1.5-0)\bar{x}_k + (1.5+0)\right] = 0.75 + 0.75\bar{x}_k .$$

Because

$$\bar{x}_0 = \cos\frac{\pi}{8} = 0.92388,$$

$$\bar{x}_1 = \cos\frac{3\pi}{8} = 0.38268,$$

$$\bar{x}_2 = \cos\frac{5\pi}{8} = -0.38268,$$

and $$\bar{x}_3 = \cos\frac{7\pi}{8} = -0.92388,$$

We have

$\tilde{x}_0 = 1.44291$, $\tilde{x}_1 = 1.03701$, $\tilde{x}_2 = 0.46299$, and $\tilde{x}_3 = 0.05709$

The Lagrange coefficient polynomials for this set of nodes are

$$\tilde{L}_0(x) = 1.8142x^3 - 2.8249x^2 + 1.0264x - 0.049728,$$
$$\tilde{L}_1(x) = -4.3799x^3 + 8.5977x^2 - 3.4026x - 0.16705,$$
$$\tilde{L}_2(x) = 4.3799x^3 - 11.112x^2 + 7.1738x - 0.37415,$$
$$\tilde{L}_3(x) = -1.8142x^3 + 5.3390x^2 - 4.7976x + 1.2568.$$

The functional values required for these polynomials are given in the last two columns of Table 3.2.1. The interpolation polynomial of degree at most 3 is

$$\tilde{P}_3(x) = 1.3811x^3 + 0.04465x^2 + 1.3031x - 0.014352.$$

Table 3.2.1

x	$f(x) = xe^x$	$\tilde{x}$	$f(\tilde{x}) = xe^x$
$x_0 = 0.0$	0.000	$\tilde{x}_0 = 1.44291$	6.10783
$x_1 = 0.5$	0.824361	$\tilde{x}_1 = 1.03701$	2.92517
$x_2 = 1.0$	2.71828	$\tilde{x}_2 = 0.46299$	0.73560
$x_3 = 1.5$	6.72253	$\tilde{x}_3 = 0.05709$	0.060444

For comparison, Table 2.2.2 lists various values of x, together with the values of $f(x)$, $P_3(x)$, and $\tilde{P}_3(x)$. It can be seen from this table that, although the error using $P_3(x)$, is less than using $\tilde{P}_3(x)$. near the middle of the table, the maximum error involved with using 1is considerably less than when using $P_3(x)$, which gives the error 0.0290. (see Figure 7)

Table 3.2.2

x	$f(x) = xe$	$P_3(x)$	$\lvert xe^x - \tilde{P}_3(x$	$\tilde{P}_3(x)$	$\lvert xe^x - \tilde{P}_3($
0.15	0.1743	0.1969	0.0226	0.1868	0.0125
0.25	0.3210	0.3435	0.0225	0.3358	0.0148
0.35	0.4967	0.5121	0.0154	0.5064	0.0097
0.65	1.245	1.233	0.012	1.231	0.014
0.75	1.588	1.572	0.016	1.571	0.017
0.85	1.989	1.976	0.013	1.974	0.015
1.15	3.632	3.650	0.018	3.644	0.012
1.25	4.363	4.391	0.028	4.382	0.019
1.35	5.208	5.237	0.029	5.224	0.016

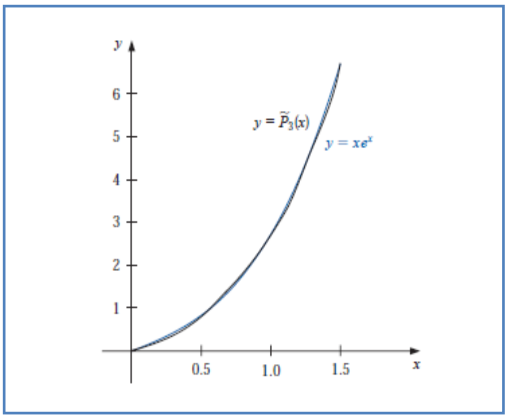

Figure 7

3.3 Reducing the Degree of Approximating Polynomials

Chebyshev polynomials can also be used to reduce the degree of n approximating polynomial with a minimal loss of accuracy . Because the Chebyshev polynomials have a minimum maximum-absolute value that is spread uniformly on an interval, they can be used to reduce of an approximation polynomial without exceeding the error tolerance.

Consider approximating an arbitrary n th-degree polynomial

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

on $[-1,1]$ with a polynomial of degree at most $n-1$. The object is choose $P_{n-1}(x)$ in $\prod_{n-1}$ so that

$$\max_{x\in[-1,1]} |P_n(x) - P_{n-1}(x)|$$

is as small as possible.

We first note that $\left(\dfrac{P_n(x) - P_{n-1}(x)}{a_n}\right)$ is a monic polynomial of degree n, so applying Theorem 3.1.1 gives

$$\max_{x\in[-1,1]} \left| \frac{1}{a_n}\left(P_n(x) - P_{n-1}(x)\right) \right| \geq \frac{1}{2^{n-1}}.$$

Equality occurs precisely when

$$\frac{1}{a_n}\left(P_n(x) - P_{n-1}(x)\right) = \tilde{T}_n(x).$$

This means that we should choose

$$P_{n-1}(x) = P_n(x) - a_n \tilde{T}_n(x),$$

and with this choice we have the minimum value of

$$\max_{x\in[-1,1]} |P_n(x) - P_{n-1}(x)| = |a_n| \max_{x\in[-1,1]} \left| \frac{1}{a_n}\left(P_n(x) - P_{n-1}(x)\right) \right| = \frac{|a_n|}{2^{n-1}}.$$

***Example 3.3.1*:**

The function $f(x) = e^x$ is approximated on the interval $[-1,1]$ by the fourth Maclaurin polynomial

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24},$$

which has truncation error

$$|R_4(x)| = \frac{\left|f^{(5)}(\xi(x))\right| |x^5|}{120} \leq \frac{e}{120} \approx 0.023, \quad \text{for} -1 \leq x \leq 1$$

Suppose that an error of 0.05 is tolerable and that we would like to reduce the degree of the approximating polynomial while staying within this bound.

The polynomial of degree 3 or less that best uniformly approximates $P_4(x)$ on $[-1,1]$ is

$$\begin{aligned} P_3(x) &= P_4(x) - a_4 \tilde{T}_4(x) \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{1}{24}\left(x^4 - x^2 + \frac{1}{8}\right) \\ &= \frac{191}{192} + x + \frac{13}{24}x^2 + \frac{1}{6}x^3. \end{aligned}$$

with this choice, we have

$$|P_4(x) - P_3(x)| = |a_4 \tilde{T}_4(x)| \leq \frac{1}{24} \cdot \frac{1}{2^3} = \frac{1}{192} \leq 0.0053.$$

Adding this error bound to the bound for the Maclaurin truncation error gives

$$0.023 + 0.0053 = 0.0283,$$

Which is within the permissible error of 0.05.

The polynomial of degree 2 or less that best uniformly approximates $P_3(x)$ on $[-1,1]$ is

$$\begin{aligned} P_2(x) &= P_3(x) - \frac{1}{6}\tilde{T}_3 \\ &= \frac{191}{192} + x + \frac{13}{24}x^2 + \frac{1}{6}x^3 - \frac{1}{6}\left(x^3 - \frac{3}{4}x\right) \\ &= \frac{191}{192} + \frac{9}{8}x + \frac{13}{24}x^2. \end{aligned}$$

However,

$$|P_3(x) - P_2(x)| = \left|\frac{1}{6}\tilde{T}_3(x)\right| = \frac{1}{6}\left(\frac{1}{2}\right)^2 = \frac{1}{24} \approx 0.042,$$

which—when added to the already accumulated error bound of 0.0283 — exceeds the tolerance of 0.05. Consequently, the polynomial of least degree that best approximates e^x on $[-1,1]$ with an error bound of less than 0.05 is

$$P_3(x) = \frac{191}{192} + x + \frac{13}{24}x^2 + \frac{1}{6}x^3.$$

Table 2.3.1 lists the function and the approximating polynomials at various points in $[-1,1]$.

Note that the tabulated entries for P_2 are well within the tolerance of 0.05, even though the error bound for $P_2(x)$ exceeded the tolerance .

Table 2.3.1

x	e^x	$P_4(x)$	$P_3(x)$	$P_2(x)$	$\lvert e^x - P_2(x)\rvert$
-0.75	0.47237	0.47412	0.47917	0.45573	0.01664
-0.25	0.77880	0.77881	0.77604	0.74740	0.03140
0.00	1.00000	1.00000	0.99479	0.99479	0.00521
0.25	1.28403	1.28402	1.28125	1.30990	0.02587
0.75	2.11700	2.11475	2.11979	2.14323	0.02623

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